1. For each linear transformation described below, determine whether or not it is invertible and, if it is invertible, find the matrix of its inverse.

(a) $P : \mathbb{R}^2 \to \mathbb{R}^2$, defined by $P(\vec{x}) = \left(\dfrac{\vec{x}\cdot\vec{a}}{\vec{a}\cdot\vec{a}}\right)\vec{a}$, where $\vec{a}$ is some fixed nonzero vector.

$P(\vec{x}) = \mathrm{proj}_{\vec{a}}(\vec{x}) \Rightarrow P$ is <u>not</u> invertible. To see why, let $\vec{b}$ be any vector perpendicular to $\vec{a}$ (other than $\vec{0}$). Then $P(\vec{b}) = \vec{0}$ but $\vec{b} \neq \vec{0}$.

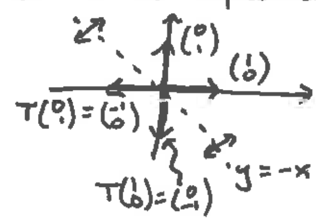
(b) $Q : \mathbb{R}^2 \to \mathbb{R}^2$, defined by letting $Q(\vec{x})$ be the result of rotating $\vec{x}$ by $\frac{7\pi}{6}$ radians about the origin.

Q is invertible! Q^{-1} rotates by $-\frac{7\pi}{6}$ radians about the origin, so the matrix of Q^{-1} is

$$\begin{pmatrix} \cos -7\pi/6 & -\sin -7\pi/6 \\ \sin -7\pi/6 & \cos -7\pi/6 \end{pmatrix} = \begin{pmatrix} -\sqrt{3}/2 & -1/2 \\ 1/2 & -\sqrt{3}/2 \end{pmatrix}$$

(c) $R : \mathbb{R}^2 \to \mathbb{R}^2$, defined by letting $R(\vec{x})$ be the result of reflecting $\vec{x}$ through the line $y = -x$.

R is invertible! $R^{-1} = R$ since R is a reflection. Its matrix is given by $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.



(d) $S : \mathbb{R}^2 \to \mathbb{R}^3$, defined by letting $S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$.

S is <u>not</u> invertible since it is not a linear endomorphism (i.e. it is a transformation $\mathbb{R}^2 \longrightarrow \mathbb{R}^3$ but $2 \neq 3$)

(e) $T : \mathbb{R}^2 \to \mathbb{R}^2$, defined by

$$T(\vec{x}) = AC^{-1}B^{-1}A^{-1}BCBAC^{-1}BA^{-1}C\vec{x}$$

where A, B and C are some invertible 2×2 matrices.

T is invertible! Its inverse is given by

$$T^{-1}(\vec{x}) = C^{-1}AB^{-1}CA^{-1}B^{-1}C^{-1}B^{-1}ABCA^{-1}$$

[This is using the general facts that $(MN)^{-1} = N^{-1}M^{-1}$ and $(M^{-1})^{-1} = M$]

2. Let A, B and C be $n \times n$ matrices. For each of the following statements, determine whether it is true or false and provide justification.

(a) If $A^k = I_n$ for some $k > 0$, then A is invertible.

True! If $k = 1$ then $A = I_n$, which is invertible.

If $k > 1$ then $A^k = A \cdot A^{k-1} = I_n$

$\Rightarrow A^{k-1} = A^{-1}$.

(b) If A is invertible then $A^k = I_n$ for some $k > 0$.

False (in general)! For example if $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

then A is invertible & $A^{-1} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$

but $A^k = \begin{pmatrix} 2^k & 0 \\ 0 & 2^k \end{pmatrix} \neq \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{= I_2}$ for any $k > 0$

(c) If the rank of A is n, then A is invertible.

True! Since A is an $n \times n$ matrix, if its rank is n then $\text{rref}(A) = I_n$, and so we can row reduce:

$$(A \mid I_n) \longrightarrow (I_n \mid A^{-1})$$

(d) It is possible that A be invertible and the equation $A\vec{x} = \vec{b}$ have infinitely many solutions.

False. Since A is an $n \times n$ matrix, in order for $A\vec{x} = \vec{b}$ to have ∞'ly many solutions, we must have $\text{rank}(A) < n$. But then $\text{rref}(A) \neq I_n$, so we cannot row reduce $(A \mid I_n) \longrightarrow (I_n \mid *)$.

(e) If there are matrices B and C such that $ABC = I_n$, then B is invertible.

True! A is invertible $\because A(BC) = I_n$

$\Rightarrow A^{-1} = BC \quad (*)$

and C is invertible $\because (AB)C = I_n$

$\Rightarrow C^{-1} = AB$

$\Rightarrow B = BCC^{-1} = A^{-1}C^{-1}$ by $(*)$

But A^{-1} & C^{-1} are invertible $\Rightarrow B$ is invertible

(and $B^{-1} = CA$).