1. For each of the following matrices, determine whether or not it is invertible and, if it is invertible, find its inverse matrix.

(a) $\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$ — invertible!

$$\left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right) \xrightarrow[(\mathrm{II})\times 2]{(\mathrm{I})\times 3} \left(\begin{array}{cc|cc} 6 & 3 & 3 & 0 \\ 6 & 4 & 0 & 2 \end{array}\right) \xrightarrow{(\mathrm{II})-(\mathrm{I})} \left(\begin{array}{cc|cc} 6 & 3 & 3 & 0 \\ 0 & 1 & -3 & 2 \end{array}\right)$$

$$\downarrow (\mathrm{I}) - 3\times(\mathrm{II})$$

$$A^{-1} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \qquad \left(\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -3 & 2 \end{array}\right) \xleftarrow[(\mathrm{I})\div 6]{} \left(\begin{array}{cc|cc} 6 & 0 & 12 & -6 \\ 0 & 1 & -3 & 2 \end{array}\right)$$

(b) $\begin{pmatrix} 2 & 1 & 0 \\ 3 & 2 & -1 \end{pmatrix}$

Not invertible $\because$ not a square matrix

(c) $\begin{pmatrix} 4 & 1 & 2 \\ 3 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix}$ Not invertible $\because$ rank < 3 :

$$\begin{pmatrix} 4 & 1 & 2 \\ 3 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix} \xrightarrow{(\mathrm{I})-(\mathrm{II})} \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix} \xrightarrow{(\mathrm{III})-(\mathrm{I})} \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\downarrow (\mathrm{II}) - 3\times(\mathrm{I})$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

(d) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ — invertible!

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[(\mathrm{III})-(\mathrm{II})]{(\mathrm{I})-(\mathrm{II})} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

2. Find a 3×2 matrix A such that

$$A \overbrace{\begin{pmatrix} 1 & -3 \\ -2 & 7 \end{pmatrix}}^{= B} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \\ 2 & -3 \end{pmatrix}$$

① Find B^{-1}:

$$\left(\begin{array}{cc|cc} 1 & -3 & 1 & 0 \\ -2 & 7 & 0 & 1 \end{array}\right) \xrightarrow{(\text{II})+2(\text{I})} \left(\begin{array}{cc|cc} 1 & -3 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{array}\right) \xrightarrow{(\text{I})+3(\text{II})} \left(\begin{array}{cc|cc} 1 & 0 & 7 & 3 \\ 0 & 1 & 2 & 1 \end{array}\right)$$

$$\Rightarrow B^{-1} = \begin{pmatrix} 7 & 3 \\ 2 & 1 \end{pmatrix}$$

$$\Rightarrow A = AI_2 = ABB^{-1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 7 & 3 \\ 2 & 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 7 & 3 \\ -12 & -5 \\ 8 & 3 \end{pmatrix}}}$$

3. For which values of b and c is the following matrix invertible?

$$\begin{pmatrix} 0 & 1 & b \\ -1 & 0 & c \\ -b & -c & 0 \end{pmatrix}$$

We compute its rank.

$$(\text{I}) \leftrightarrow (\text{II}): \begin{pmatrix} -1 & 0 & c \\ 0 & 1 & b \\ -b & -c & 0 \end{pmatrix} \xrightarrow{(\text{I})\times(-1)} \begin{pmatrix} 1 & 0 & -c \\ 0 & 1 & b \\ -b & -c & 0 \end{pmatrix} \xrightarrow[+c(\text{II})]{(\text{III})+b(\text{I})} \begin{pmatrix} 1 & 0 & -c \\ 0 & 1 & b \\ 0 & 0 & \underbrace{\boxed{bc-bc}}_{=0} \end{pmatrix}$$

The rank of $\begin{pmatrix} 0 & 1 & b \\ -1 & 0 & c \\ -b & -c & 0 \end{pmatrix}$ is 2 for all values of b, c, so it is <u>never</u> invertible!

4. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix.

Prove that A is invertible if and only if $ad = bc$, and that in this case A^{-1} is defined as follows.

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

We try to find A^{-1}:

$$\left(\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right) \xrightarrow[(\mathrm{II})\times a]{(\mathrm{I})\times c} \left(\begin{array}{cc|cc} ac & bc & c & 0 \\ ac & ad & 0 & a \end{array}\right) \xrightarrow{(\mathrm{II})-(\mathrm{I})} \left(\begin{array}{cc|cc} ac & bc & c & 0 \\ 0 & ad-bc & -c & a \end{array}\right)$$

If $ad - bc = 0$ then rank $\begin{pmatrix} a & b \\ c & d \end{pmatrix} < 2 \Rightarrow$ $\boxed{\text{not investible}}$

So assume $ad \neq bc$ ($\Rightarrow ad - bc \neq 0$). Then:

proposed value

$$A^{-1}A = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ad-bc & bd-bd \\ ac-ac & ad-bc \end{pmatrix}\frac{1}{ad-bc}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\Rightarrow A^{-1}$ is as claimed.

5. Let A be an *upper-triangular* $n \times n$ matrix, i.e. such that $a_{ij} = 0$ if $i > j$. Prove that A is invertible if and only if $a_{11} \times a_{22} \times \cdots \times a_{nn} \neq 0$.

$$A = \begin{pmatrix} a_{11} & * & \cdots & * \\ 0 & a_{22} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \longleftarrow$$

This is in ref if each $a_{ii} \neq 0 \Rightarrow$ rank $= n$ $\Rightarrow$ invertible.

If at least one $a_{ii} = 0$, let i be the greatest such that $a_{ii} = 0$ (so $a_{jj} \neq 0$ for all $i < j \leq n$) Then

$$A = \begin{pmatrix} \cdots & * & \cdots \\ 0\,0 \cdots & a_{ii}=0 & \cdots\ a_{in} \\ -0- & \neq 0 & \cdots\ * \\ -\!-\,0\,-\!- & & \neq 0 \end{pmatrix}$$

By subtracting $\frac{a_{ij}}{a_{jj}} \times$ row j from row i, get $(0\ 0\ \cdots\ 0\ 0)$ in row i $\longrightarrow$ rank $< n$ $\longrightarrow$ not invertible.