1. For each of the following products of matrices, either compute its value, or explain why it is not defined.

(a) $$\underbrace{\begin{pmatrix} 2 & 2 \\ 1 & -1 \\ 3 & 1 \end{pmatrix}}_{3\times \underline{2}} \underbrace{\begin{pmatrix} 0 & 1 \\ 3 & 1 \\ -2 & -2 \end{pmatrix}}_{\underline{3}\times 2}$$

not defined

(b) $$\begin{pmatrix} 2 & 2 \\ 1 & -1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 0 & 3 & -2 \\ 1 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 2\cdot 0 + 2\cdot 1 & 2\cdot 3 + 2\cdot 1 & 2\cdot(-2) + 2\cdot(-2) \\ 1\cdot 0 + (-1)\cdot 1 & 1\cdot 3 + (-1)\cdot 1 & 1\cdot(-2) + (-1)\cdot(-2) \\ 3\cdot 0 + 1\cdot 1 & 3\cdot 3 + 1\cdot 1 & 3\cdot(-2) + 1\cdot(-2) \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 8 & -8 \\ -1 & 2 & 0 \\ 1 & 10 & -8 \end{pmatrix}$$

(c) $$\begin{pmatrix} 2 & 1 & 3 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 3 & 1 \\ -2 & -2 \end{pmatrix} = \begin{pmatrix} 2\cdot 0 + 1\cdot 3 + 3\cdot(-2) & 2\cdot 1 + 1\cdot 1 + 3\cdot(-2) \\ 2\cdot 0 + (-1)\cdot 3 + 1\cdot(-2) & 2\cdot 1 + (-1)\cdot 1 + 1\cdot(-2) \end{pmatrix}$$

$$= \begin{pmatrix} -3 & -3 \\ -5 & -1 \end{pmatrix}$$

(d) $$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{} \begin{pmatrix} 23 & 56 & 89 \\ 13 & 24 & 35 \\ 77 & 44 & 11 \end{pmatrix} = \begin{pmatrix} 23 & 56 & 89 \\ 13 & 24 & 35 \\ 77 & 44 & 11 \end{pmatrix}$$

This is the 3×3 identity matrix

(e) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Geometrically, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is reflection through the line $y = x$.

Doing a reflection twice is the same as doing nothing!

(f) $\begin{pmatrix} b_1 & b_2 & \cdots & b_n \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = b_1 a_1 + b_2 a_2 + \cdots + b_n a_n = \vec{b} \cdot \vec{a}$

(g) $\underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}}_{n \times 1} \underbrace{\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}}_{n \times 1}$

Undefined

Note this is not the same as

$$\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

dot product

(h) $\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix} = \begin{pmatrix} b_1 a_1 & b_1 a_2 & \cdots & b_1 a_n \\ b_2 a_1 & b_2 a_2 & \cdots & b_2 a_n \\ \vdots & \vdots & \ddots & \vdots \\ b_n a_1 & b_n a_2 & \cdots & b_n a_n \end{pmatrix}$

Again, this is not the same as

$$\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \cdot \begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix}$$

dot product

2. Four 2×2 matrices A, B, C and D are defined as follows. Which pairs commute?

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad D = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$$

A: scaling by a factor of 2

B: rotation by $\frac{\pi}{2}$ radians

C: reflection in x-axis

D: rotation by $\frac{\pi}{3}$ radians

Pair	Commutes?	Why?
A, B	Yes	$A\vec{v} = 2\vec{v}$ for all $\vec{v}$, hence $AT(\vec{v}) = 2T(\vec{v}) = T(2\vec{v}) = T(A\vec{v})$ for all linear transformations $T: \mathbb{R}^2 \to \mathbb{R}^2$ (in particular when T is multiplication by B, C or D)
A, C	Yes	(same as above)
A, D	Yes	(same as above)
B, C	No	$BC = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = CB$
B, D	Yes	BD and DB both represent rotation by $\frac{\pi}{2} + \frac{\pi}{3} = \frac{5\pi}{6}$ radians
C, D	No	$CD = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix} \neq \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} = DC$

3. Let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that rotates each vector by θ radians anticlockwise about the origin, and let A_θ be the associated 2×2 matrix.

$$A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Using the fact that $R_\alpha \circ R_\beta = R_{\alpha+\beta}$, find expressions for $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$ in terms of $\sin\alpha$, $\cos\alpha$, $\sin\beta$ and $\cos\beta$.

Since $R_\alpha \circ R_\beta = R_{\alpha+\beta}$ we have $A_\alpha A_\beta = A_{\alpha+\beta}$.

$$\Rightarrow \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix}$$

$$= \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{pmatrix}$$

$$\Rightarrow \begin{cases} \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \end{cases}$$