1. For each of the following functions T, determine whether or not it is linear. If it is linear, find a matrix A such that $T(\vec{v}) = A\vec{v}$ for all vectors $\vec{v}$.

(a) $T : \mathbb{R}^2 \to \mathbb{R}$, defined by $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1^2 + x_2^2$

Not linear, e.g. $T\left(2 \times \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = T\begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2^2 + 0^2 = 4$

but $2\,T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2(1^2 + 0^2) = 2$

(b) $T : \mathbb{R}^2 \to \mathbb{R}^2$, defined by $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 - x_1 \\ x_1 + x_2 \end{pmatrix}$

Linear. $T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

So letting $A = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ gives $A\vec{v} = T(\vec{v})$ for all $\vec{v}$.

(c) [Bretscher §2.1 Q4] $T : \mathbb{R}^3 \to \mathbb{R}^4$, defined by the linear system

$$\begin{cases} y_1 = 9x_1 + 3x_2 - 3x_3 \\ y_2 = 2x_1 - 9x_2 + x_3 \\ y_3 = 4x_1 - 9x_2 - 2x_3 \\ y_4 = 5x_1 + x_2 + 5x_3 \end{cases}$$

(i.e. T is defined by $\vec{y} = T(\vec{x})$, with $\vec{y}$ defined in terms of $\vec{x}$ according to these four equations)

Linear. $T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 9 \\ 2 \\ 4 \\ 5 \end{pmatrix}$, $T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -9 \\ -9 \\ 1 \end{pmatrix}$, $T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ -2 \\ 5 \end{pmatrix}$

$\Rightarrow A = \begin{pmatrix} 9 & 3 & -3 \\ 2 & -9 & 1 \\ 4 & -9 & -2 \\ 5 & 1 & 5 \end{pmatrix}$ is the associated matrix.

(d) [Bretscher §2.1 Q4, modified]

$$\begin{cases} y_1 = 9x_1 + 3x_2 - 3x_3 + 2 \\ y_2 = 2x_1 - 9x_2 + x_3 \\ y_3 = 4x_1 - 9x_2 - 2x_3 - 1 \\ y_4 = 5x_1 + x_2 + 5x_3 + 1 \end{cases}$$

Not linear, e.g.

$T\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) = T\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \end{pmatrix}$

but $T\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + T\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -2 \\ 2 \end{pmatrix}$

2. Given a vector $\vec{a}$ in $\mathbb{R}^n$, find a matrix A such that $A\vec{v} = \vec{a} \cdot \vec{v}$ for all $\vec{v}$ in $\mathbb{R}^n$.

Let $\vec{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$. Then

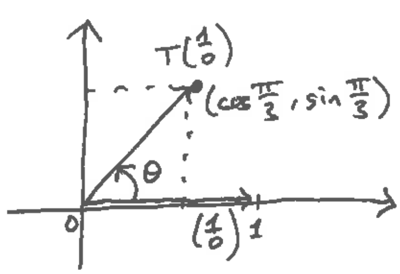
$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = a_1 v_1 + \cdots + a_n v_n = \vec{a} \cdot \vec{v}$$

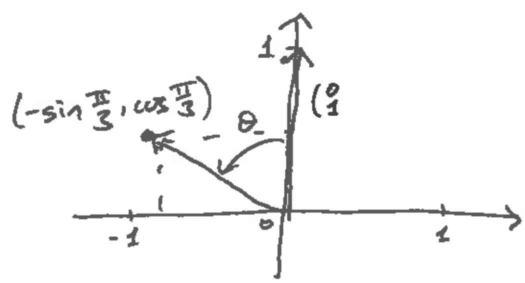
so we can take A to be the row vector whose entries are the same as $\vec{a}$.

3. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be any linear transformation. Find $T(\vec{0})$, where $\vec{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ is the *zero vector*.

$$\underbrace{T(\vec{0})}_{\substack{\text{n-dimensional} \\ \text{zero vector}}} = T(0\vec{e}_1) \underbrace{=}_{\text{by linearity}} 0\, T(\vec{e}_1) = \underbrace{\vec{0}}_{\substack{\text{m-dimensional} \\ \text{zero vector}}}$$

4. Find the matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ which rotates each vector $\vec{v}$ by $\frac{\pi}{3}$ radians about the origin.





$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos \pi/3 \\ \sin \pi/3 \end{pmatrix} \qquad T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin \pi/3 \\ \cos \pi/3 \end{pmatrix}$$

So the associated matrix is

$$\begin{pmatrix} \cos \pi/3 & -\sin \pi/3 \\ \sin \pi/3 & \cos \pi/3 \end{pmatrix} = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$$