1. For each of the following expressions, either compute its value, or explain why it is not defined.

(a) $\begin{pmatrix}1\\2\\3\end{pmatrix}\cdot\begin{pmatrix}3\\2\\1\end{pmatrix}$ $\quad 1\times 3+2\times 2+3\times 1 = 3+4+3 = 10$

(b) $\begin{pmatrix}1\\2\\3\end{pmatrix}\cdot\begin{pmatrix}3 & 2 & 1\end{pmatrix}$ $\quad 1\times 3+2\times 2+3\times 1 = 3+4+3 = 10$

(c) $2\begin{pmatrix}1\\2\\3\end{pmatrix}-\begin{pmatrix}3\\2\\1\end{pmatrix} = \begin{pmatrix}2\\4\\6\end{pmatrix}-\begin{pmatrix}3\\2\\1\end{pmatrix} = \begin{pmatrix}-1\\2\\5\end{pmatrix}$

(d) $2\begin{pmatrix}1\\2\\3\end{pmatrix}-\begin{pmatrix}3 & 2 & 1\end{pmatrix}$ Not defined — can't have linear combination of column & row vector.

(e) $\begin{pmatrix}1 & 2\\3 & 2\\1 & 0\end{pmatrix}\begin{pmatrix}-1\\0\\2\end{pmatrix}$ Not defined — $A\vec{v}$ is only defined when A is $m\times n$ & $\vec{v}$ is $n\times 1$, but here A is 3×2 and $\vec{v}$ is 2×1.

(f) $\begin{pmatrix}1 & 3 & 1\\2 & 2 & 0\end{pmatrix}\begin{pmatrix}-1\\0\\2\end{pmatrix} = \begin{pmatrix}1\times(-1)+3\times 0+1\times 2\\2\times(-1)+2\times 0+0\times 2\end{pmatrix} = \begin{pmatrix}-1+0+2\\-2+0+0\end{pmatrix} = \begin{pmatrix}1\\-2\end{pmatrix}$

(g) $\begin{pmatrix}1 & 0 & 1\\0 & -2 & -1\\1 & 2 & 3\end{pmatrix}\begin{pmatrix}2\\3\\-2\end{pmatrix} = \begin{pmatrix}1\times 2+0\times 3+1\times(-2)\\0\times 2-2\times 3-1\times(-2)\\1\times 2+2\times 3+3\times(-2)\end{pmatrix} = \begin{pmatrix}2+0-2\\0-6+2\\2+6-6\end{pmatrix} = \begin{pmatrix}0\\-4\\2\end{pmatrix}$

2. Express the following linear system in the form $A\vec{x} = \vec{b}$.

$$\begin{cases} 2x_1 - x_2 + 3x_3 - x_4 + 2x_5 = 0 \\ x_3 + x_4 = 2 \\ -x_1 - x_2 + 5x_4 = -1 \\ 3x_1 - x_5 = 4 \end{cases}$$

[Your answer is called the **matrix form** of the system.]

$$\underbrace{\begin{pmatrix} 2 & -1 & 3 & -1 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ -1 & -1 & 0 & 5 & 0 \\ 3 & 0 & 0 & 0 & -1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} 0 \\ 2 \\ -1 \\ 4 \end{pmatrix}}_{\vec{b}}$$

3. Explain why every vector in $\mathbb{R}^3$ can be expressed as a linear combination of the vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$, where

$$\vec{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}, \quad \vec{w} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$$

Let $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be a vector in $\mathbb{R}^3$. If it is to be a linear combination of $\vec{u}, \vec{v}, \vec{w}$, that is to say there exist scalars x, y, z such that $x\vec{u} + y\vec{v} + z\vec{w} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, i.e.

$$x\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + y\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + z\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

This is equivalent to the linear system

$$\begin{cases} x + 2y + 3z = a \\ x + 2y = b \\ x = c \end{cases}$$

having a solution. But this system has a unique solution for all values of a, b, c since its coefficient matrix $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ has rank 3 and is a 3×3 matrix.