Math 290-1 Class 2

Monday 1st October 2018

Updated: Tuesday 2nd October 2018

Gauss–Jordan elimination

An $m \times n$ **matrix** is a grid of numbers with m rows and n columns.

$$\underset{2\times 3\text{ matrix}}{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}} \qquad \underset{3\times 2\text{ matrix}}{\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}} \qquad \underset{4\times 4\text{ matrix}}{\begin{pmatrix} 1 & -1 & 1 & -1 \\ -2 & 2 & -2 & 2 \\ 3 & -3 & 3 & -3 \\ 4 & -4 & 4 & -4 \end{pmatrix}}$$

The entries of an $m \times n$ matrix A will be written a_{ij}, where i represents the **row** ($1 \leqslant i \leqslant m$) and j represents the **column** ($1 \leqslant j \leqslant n$) of the entry.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \end{pmatrix}$$

An $m \times 1$ matrix is called a **column vector** (or just **vector**), a $1 \times n$ is called a **row vector**, and an $n \times n$ matrix is called a **square matrix**. $\mathbb{R}^n$ is the **vector space** of column vectors with n entries.

$$\underset{\text{column vector}}{\begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}} \qquad \underset{\text{row vector}}{\begin{pmatrix} 2 & 5 & 8 & 11 & 14 \end{pmatrix}} \qquad \underset{\text{square matrix}}{\begin{pmatrix} 1 & -2 \\ -3 & 4 \end{pmatrix}}$$

An **augmented matrix** is a horizontal concatenation of an $m \times n$ matrix with an $m \times k$ matrix (usually $k = 1$ or $k = m$). When $k = 1$, the augmented matrix can be used to represent a linear system.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \end{array}\right) \quad \text{represents} \quad \begin{cases} x + 2y + 3z = 4 \\ 4x + 5y + 6z = 7 \end{cases}$$

We can do elementary row operations to augmented matrices just like we did for linear systems. The goal is to put the left-hand matrix in **reduced row-echelon form (rref)**.

$$\left(\begin{array}{ccccc|c} \boxed{1} & -2 & 0 & 0 & 2 & 2 \\ 0 & 0 & \boxed{1} & 0 & 1 & 3 \\ 0 & 0 & 0 & \boxed{1} & -2 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{cases} \boxed{x_1} - 2x_2 + 4x_3 \qquad + 2x_5 = 2 \\ \qquad\qquad \boxed{x_3} \qquad + x_5 = 3 \\ \qquad\qquad\qquad \boxed{x_4} - 2x_5 = -4 \\ \qquad\qquad\qquad\qquad 0 = 0 \end{cases}$$

Setting non-leading variables (in this case x_2 and x_5) equal to parameters and solving for the leading variables (in this case x_1, x_3 and x_4) provides a parametrised solution to the linear system.

Examples

1. [Bretscher §1.2 Q18] Determine which of the following (possibly augmented) matrices are in reduced row-echelon form; if it isn't, say why not.

(a) $\begin{pmatrix} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ $\longleftarrow$ Not in rref: circled 1 is below a leading 1 (pivot)

(b) $\begin{pmatrix} 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ $\longleftarrow$ This <u>is</u> in rref

(c) $\left(\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right)$ $\longleftarrow$ Not in rref: row 2 has no pivots but row 3 has a pivot

(d) $\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \end{pmatrix}$ $\longleftarrow$ This <u>is</u> in rref

(e) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\longleftarrow$ This <u>is</u> in rref

(f) $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 3 \end{array}\right)$ $\longleftarrow$ Not in rref: circled 3 is the leading nonzero entry but is not a 1.

2. [Bretscher §1.2 Q6, modified] Consider the following linear system.

$$\begin{cases} x_1 - 7x_2 \qquad\qquad + \ x_5 = 3 \\ \qquad\qquad x_3 \qquad - 2x_5 = 2 \\ \qquad\qquad\qquad x_4 + \ x_5 = 1 \end{cases}$$

Write down the augmented matrix representing the system.

$$\left(\begin{array}{ccccc|c} 1 & -7 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & -2 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right)$$

This is in rref!

Find all solutions to the system.

Free variables: x_2, x_5.

Let $x_2 = s$ and $x_5 = t$. Then:

$$x_1 = 3 + 7x_2 - x_5 = 3 + 7s - t$$
$$x_3 = 2 + 2x_5 = 2 + 2t$$
$$x_4 = 1 - x_5 = 1 - t$$

So the solutions to the system are

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 3 + 7s - t \\ s \\ 2 + 2t \\ 1 - t \\ t \end{pmatrix}$$

for all real values of s and t.

3. [Bretscher §1.2 Q11, modified] Use Gauss–Jordan elimination to solve the following linear system.

$$\begin{cases} x_1 \qquad\quad 2x_3 + 4x_4 = -8 \\ x_2 - 3x_3 - x_4 = 6 \\ 3x_1 + 4x_2 - 6x_3 + 8x_4 = 0 \\ -x_2 + 3x_3 + 4x_4 = -12 \end{cases}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 3 & 4 & -6 & 8 & 0 \\ 0 & -1 & 3 & 4 & -12 \end{array}\right)$$

Step 1 Circled 1 is a pivot, so we clear the remaining nonzero entries in its column.

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 0 & 4 & -12 & -4 & 24 \\ 0 & -1 & 3 & 4 & -12 \end{array}\right) \quad (\mathrm{III}) - 3\times(\mathrm{I})$$

Step 2 Circled 1 is a pivot, so we clear the remaining nonzero entries in its column

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & -6 \end{array}\right) \quad \begin{array}{l} (\mathrm{III}) - 4\times(\mathrm{II}) \\ (\mathrm{IV}) + (\mathrm{II}) \end{array}$$

Step 3 Circled 3 must be turned into a 1 to become a pivot

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right) \quad (\mathrm{IV}) \div 3$$

Step 4 Circled 1 is a pivot, so we clear the remaining nonzero entries in its column

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right) \quad \begin{array}{l} (\mathrm{I}) - 4\times(\mathrm{IV}) \\ (\mathrm{II}) + (\mathrm{IV}) \end{array}$$

Step 5 Move zero rows to bottom + make sure pivots are in the right order

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 4 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \left.\begin{array}{l} (\mathrm{IV}) \\ (\mathrm{III}) \end{array}\right\} \text{swap}$$

Free variable: x_3. Let $x_3 = t$.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2t \\ 4+3t \\ t \\ -2 \end{pmatrix}$$ is a solution for all real values of t.