Examples

1. Just by looking at the following linear systems of 2 equations in 2 variables, indicate next to each system whether you expect it to have no solution, a unique solution, or infinitely many solutions..

$$\begin{cases} x + 2y = 1 \\ 2x + 3y = 1 \end{cases}$$

Number of solutions: none one infinitely many

Explanation: Lines are not parallel (Slopes are $-1/2$ and $-2/3$)

$$\begin{cases} x + 2y = 1 \\ 2x + 4y = 1 \end{cases}$$

Number of solutions: none one infinitely many

Explanation: Lines are parallel but not the same (Slope $= -1/2$)

$$\begin{cases} x + 2y = 1 \\ 2x + 4y = 2 \end{cases}$$

Number of solutions: none one infinitely many

Explanation: Both equations define the same line (Slope $= -1/2$)

$$\begin{cases} 3x - 2y = 4 \\ 2x - y = 1 \end{cases}$$

Number of solutions: none one infinitely many

Explanation: Lines are not parallel (Slopes are $3/2$ and 2)

Warning! Order of variables is swapped!

$$\begin{cases} x - 2y = -1 \\ 2y - x = 0 \end{cases}$$

Number of solutions: none one infinitely many

Explanation: Lines are parallel (Slope $= 1/2$)

$$\begin{cases} -6x + 2y = 4 \\ 3x - y = -4 \end{cases}$$

Number of solutions: none one infinitely many

Explanation: Lines are parallel (Slope $= 1/3$)

2. [Bretscher §1.1 Q9] Find all the solutions to the following linear system and give a geometric account for your answer.

$$\begin{cases} x + 2y + 3z = 1 \\ 3x + 2y + z = 1 \\ 7z + 2y - 3z = 1 \end{cases}$$

$$\Rightarrow \begin{cases} x + 2y + 3z = 1 & \\ - 4y - 8z = -2 & (\mathrm{II}) - 3\times(\mathrm{I}) \\ - 12y - 24z = -6 & (\mathrm{III}) - \cancel{7}\times(\mathrm{I}) \end{cases}$$

$$\Rightarrow \begin{cases} x + 2y + 3z = 1 & \\ y + 2z = \frac{1}{2} & (\mathrm{II}) \div (-4) \\ y + 2z = \frac{1}{2} & (\mathrm{III}) \div (-12) \end{cases}$$

$$\Rightarrow \begin{cases} x + 2y + 3z = 1 & \\ y + 2z = \frac{1}{2} & \\ 0 = 0 & (\mathrm{III}) - (\mathrm{II}) \end{cases}$$

The planes meet at a line $\Rightarrow$ infinitely many solutions.

Let $z = t$. (The variable t is called a <u>parameter</u>.)

$$\Rightarrow y = \tfrac{1}{2} - 2z = \tfrac{1}{2} - 2t$$

$$\Rightarrow x = 1 - 2y - 3z = 1 - 2\left(\tfrac{1}{2} - 2t\right) - 3t$$
$$= 1 - 1 + 4t - 3t$$
$$= t$$

So the solutions are $(x, y, z) = \left(t, \tfrac{1}{2} - 2t, t\right)$ for all real values of t.

3. [Bretscher §1.1 Q18] Let a, b and c be given. Find all the solutions to the following linear system in terms of a, b and c.

$$\begin{cases} x + 2y + 3z = a \\ x + 3y + 8z = b \\ x + 2y + 2z = c \end{cases}$$

$$\Rightarrow \left\{ \begin{array}{rrrcl} x & +\,2y & +\,3z & = & a \\ & y & +\,5z & = & -a + b \\ & & -z & = & -a \qquad + c \end{array} \right. \quad \left| \begin{array}{l} \\ (\mathrm{II}) - (\mathrm{I}) \\ (\mathrm{III}) - (\mathrm{I}) \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{rrrcl} x & +\,2y & +\,3z & = & a \\ & y & & = & -6a + b + 5c \\ & & z & = & a \qquad - c \end{array} \right. \quad \left| \begin{array}{l} \\ (\mathrm{II}) + 5\times(\mathrm{III}) \\ (\mathrm{III})\times(-1) \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{rrrcl} x & +\,2y & & = & -2a \qquad + 3c \\ & y & & = & -6a + b + 5c \\ & & z & = & a \qquad - c \end{array} \right. \quad \left| \begin{array}{l} (\mathrm{I}) - 3\times(\mathrm{III}) \\ \\ \\ \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{rrrcl} x & & & = & 10a - 2b - 7c \\ & y & & = & -6a + b + 5c \\ & & z & = & a \qquad - c \end{array} \right. \quad \left| \begin{array}{l} (\mathrm{I}) - 2\times(\mathrm{II}) \\ \\ \\ \end{array} \right.$$

So the (unique) solution is $(x, y, z) = (10a - 2b - 7c,\ -6a + b + 5c,\ a - c)$

[For future reference This tells us that the inverse of the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 8 \\ 1 & 2 & 2 \end{pmatrix}$ is $\begin{pmatrix} 10 & -2 & -7 \\ -6 & 1 & 5 \\ 1 & 0 & -1 \end{pmatrix}$.]

4. [Bretscher §1.1 Q21] The sums of any two of three real numbers are 24, 28 and 30. Find these three numbers.

Let the real numbers (in some order) be x, y, z.

$$\begin{cases} x + y \phantom{{}+z} = 24 \\ \phantom{x+{}} y + z = 28 \\ x \phantom{{}+y} + z = 30 \end{cases}$$

$$\Rightarrow \begin{cases} x + y = 24 \\ y + z = 28 \\ -y + z = 6 & (\mathrm{III}) - (\mathrm{I}) \end{cases}$$

$$\Rightarrow \begin{cases} x + y = 24 \\ y + z = 28 \\ 2z = 34 & (\mathrm{III}) + (\mathrm{II}) \end{cases}$$

$$\Rightarrow \begin{cases} x + y = 24 \\ y + z = 28 \\ z = 17 & (\mathrm{III}) \div 2 \end{cases}$$

$$\Rightarrow \begin{cases} x + y = 24 \\ y = 11 & (\mathrm{II}) - (\mathrm{III}) \\ z = 17 \end{cases}$$

$$\Rightarrow \begin{cases} x = 13 & (\mathrm{I}) - (\mathrm{II}) \\ y = 11 \\ z = 17 \end{cases}$$

So the numbers are 11, 13 and 17.