The Geometry of Divisors on Matroids

by

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Abstract

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Matroids are combinatorial abstractions of hyperplane arrangements, and have been a bridge for fruitful interactions between combinatorics and algebraic geometry. In particular, the recent development of the Hodge theory of matroids in [AHK18] showed that the Chow ring of a matroid satisfies properties enjoyed by cohomology rings of smooth complex projective varieties. Namely, these are the Poincaré duality property, the hard Lefschetz property, and the Hodge-Riemann relations. The validity of these properties resolved several major conjectures in matroid theory.

In this thesis, we introduce a presentation of the Chow ring of a matroid by a new set of generators, called "simplicial generators." These generators are analogous to nef divisors on projective toric varieties, and admit a combinatorial interpretation via the theory of matroid quotients. Using this combinatorial interpretation, we (i) produce a bijection between a monomial basis of the Chow ring and a relative generalization of Schubert matroids, (ii) recover the Poincaré duality property, (iii) give a formula for the volume polynomial, which we show is log-concave in the positive orthant, and (iv) recover the validity of Hodge-Riemann relations in degree 1, which is the part of the Hodge theory of matroids that currently accounts for all combinatorial applications of [AHK18]. Our work avoids the use of "flips," which is the key technical tool employed in [AHK18]. We also apply the tools developed here to study two particular divisor classes motivated by the geometry of wonderful compactifications of hyperplane arrangement complements.

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Chapter 1

Introduction

> "It is as if one were to condense all trends of present day mathematics onto a single finite structure, a feat that anyone would a priori deem impossible, were it not for the fact that matroids do exist."
>
> – Gian-Carlo Rota, Foreword in [Kun86].

Matroids are combinatorial objects that capture the essence of linear independence. Because they admit diverse interpretations, matroids appear throughout mathematics. For example, matroids arise as graphs in discrete mathematics and as diminishing-return functions in optimization. In particular, the interpretation of matroids as hyperplane arrangements has led to fruitful interactions between algebraic geometry and matroid theory. A notable example is the development of the Hodge theory of matroids [AHK18], a breakthrough that resolved many long-standing conjectures in matroid theory.

The central object in the Hodge theory of matroids is the Chow ring $A^\bullet(M)$ of a matroid M, which can be considered as a matroid theoretic analogue of the cohomology ring of a smooth manifold. The authors of [AHK18] showed that the Chow ring $A^\bullet(M)$ satisfies properties enjoyed by the cohomology rings of smooth complex projective varieties. Namely, these are the Poincaré duality property, the hard Lefschetz property, and the Hodge-Riemann relations, which together form the "Kähler package." The validity of these properties for Chow rings of matroids has far-reaching implications, including resolutions of major conjectures in matroid theory and applications to computer science [Huh18a; HW17; HSW18; AOV18].

We present a new approach to studying Chow rings of matroids. The key idea behind our approach is to establish combinatorial counterparts of two particular phenomena in algebraic geometry. The two phenomena are described in §1.1, along with tables summarizing their combinatorial counterparts. While we apply this idea only in the context of Chow rings of matroids in this thesis, we speculate that our approach will be useful in contexts of other combinatorial analogues of Hodge theory.

In Chapter 2, we review tropical intersection theory, which is a suitable language for establishing combinatorial counterparts of the two geometric phenomena, and review relevant notions about Chow rings of matroids. We assume no knowledge of tropical geometry, but assume familiarity with the fundamentals of matroid theory. A brief account of matroids can be found in Appendix A.

In Chapter 3, we illuminate some structural properties of Chow rings of matroids via the theory of matroid quotients. The key result here is that under a new presentation of the Chow ring of a matroid, the variables now carry a combinatorial meaning as an operation in matroid theory known as principal truncations (Theorem 3.2.3).

In Chapter 4, we use our structural understanding of Chow rings of matroids from the previous chapter to recover the Poincaré duality property for Chow rings of matroids. Unlike its proof in [AHK18], our proof is not inductive.

In Chapter 5, we give a formula for volume polynomials of Chow rings of matroids, and we further show that the volume polynomials are log-concave by showing that they are Lorentzian in the sense of [BH19].

In Chapter 6, we give a simplified proof of the combinatorially relevant portion of the Hodge theory of matroids in [AHK18]. Our proof avoids using a technical tool called "flips" in [AHK18]. Thus, in contrast to the proof in [AHK18], our proof involves only classical Bergman fans associated to matroids and takes the form of a single induction instead of a double induction.

In Chapter 7, we study the properties of some geometrically distinguished divisors on a matroid M, motivated by the geometry of wonderful compactifications of hyperplane arrangement complements.

Chapters 2 through 6 are reproduced from [BES19], whereas Chapter 7 is all new.

1.1 Two motivating geometric phenomena

The first phenomenon is a simple observation.

Phenomenon I. Let X be a projective variety whose Chow ring $A^\bullet(X)$ is well-understood. More precisely, suppose that $A^\bullet(X)$ is generated by the degree one elements $A^1(X)$, and suppose further that we have a distinguished collection of base-point-free divisor classes $\zeta_1, \dots, \zeta_s \in A^1(X)$ that generate $A^1(X)$. Let $Y \subset X$ be a subvariety whose Chow ring $A^\bullet(Y)$ is an interest of study. If the pullback map $A^\bullet(X) \to A^\bullet(Y)$ is surjective, then[1]

$$A^\bullet(Y) \simeq A^\bullet(X)/\operatorname{ann}([Y]) \quad \text{where} \quad \operatorname{ann}([Y]) := \{\xi \in A^\bullet(X) \mid \xi \cdot [Y] = 0\}.$$

[1] Technically, one needs few additional assumptions on X and Y for the relation to hold. In our case, we may assume that rational and numerical equivalence coincide for these varieties, which is a feature shared by all varieties that inspire the combinatorics of this thesis. See the first footnote in §2.1.

In other words, the principal ideal $\langle [Y] \rangle$ of $A^\bullet(X)$, considered as a ring with the multiplication $\xi[Y] \cdot \xi'[Y] = (\xi \cdot \xi')[Y]$, is naturally isomorphic to the Chow ring $A^\bullet(Y)$. Thus, the study of $A^\bullet(Y)$ reduces to the study of intersection products of Y with divisor classes $\zeta_1, \ldots, \zeta_s$, and moreover, these intersection classes can be described by hyperplane pullbacks of the maps $|\zeta_i| : Y \to \mathbb{P}^{m_i}$ (which exist because $\zeta_1, \ldots, \zeta_s$ are base-point-free).

Because base-point-free divisors are nef, and because the pullbacks of nef divisor classes along closed embeddings are again nef, this first observation is particularly useful in conjunction with the following second phenomenon.

Phenomenon II (Inequalities of Hodge type)**.** [Laz04, §1.6.A] Let $\zeta_1, \ldots, \zeta_s \in A^1(X)$ be nef divisor classes on a smooth projective variety X of dimension d. Let $\int_X : A^d(X) \to \mathbb{Z}$ be the degree map, and consider the polynomial $\mathrm{vol}(\underline{t}) \in \mathbb{Z}[t_1, \ldots, t_s]$ defined by

$$\mathrm{vol}(t_1, \ldots, t_s) := \int_X (t_1\zeta_1 + \cdots + t_s\zeta_s)^d.$$

Then the polynomial $\mathrm{vol}(\underline{t})$, as a real-valued function on $\mathbb{R}^s$, is nonnegative on the orthant $\mathbb{R}^s_{\geq 0}$, and its logarithm, as a function $\log \mathrm{vol} : \mathbb{R}^s_{\geq 0} \to \mathbb{R} \cup \{-\infty\}$, is concave.

Objects in Phenomenon I	Their counterparts for a matroid M on $\{0, \ldots, n\}$
The variety X and its Chow ring $A^\bullet(X)$ that is well-understood.	The braid fan Σ_{A_n} and its Chow cohomology ring $A^\bullet(\Sigma_{A_n})$. Equivalently, its toric variety X_{A_n} also known as the permutohedral variety, and $A^\bullet(X_{A_n})$.
The distinguished base-point-free divisor classes $\zeta_1, \ldots, \zeta_s$ of $A^1(X)$.	Divisor classes $h_S \in A^1(\Sigma_{A_n})$, one for each nonempty subset S of $\{0, \ldots, n\}$, corresponding to (negative) standard simplices in $\mathbb{R}^{n+1}$.
The subvariety $Y \subset X$ of interest.	The Bergman fan Σ_M of M, a subfan of Σ_{A_n}. When M has a realization $\mathscr{R}(M)$, the wonderful compactification $Y_{\mathscr{R}(M)} \subset X_{A_n}$.
The Chow ring $A^\bullet(Y)$ of Y.	The Chow ring $A^\bullet(M)$ of M, which is the Chow cohomology ring $A^\bullet(\Sigma_M)$ of Σ_M. When M has a realization $\mathscr{R}(M)$, the Chow ring $A^\bullet(Y_{\mathscr{R}(M)})$ of $Y_{\mathscr{R}(M)}$.
The algebraic cycle $[Y] \in A^\bullet(X)$.	The Bergman class $\Delta_M \in \mathrm{MW}_\bullet(\Sigma_{A_n})$ of M, which is a Minkowski weight on Σ_{A_n}. When M has a realization $\mathscr{R}(M)$, the algebraic cycle $[Y_{\mathscr{R}(M)}] \in A^\bullet(X_{A_n})$.

Table 1.1: Combinatorial counterparts of the objects in Phenomenon I

Features in Phenomenon I	Their counterparts for a matroid M on $\{0,\dots,n\}$
If $A^\bullet(X) \twoheadrightarrow A^\bullet(Y)$, then one has an isomorphism $$A^\bullet(Y) \simeq A^\bullet(X)/\operatorname{ann}([Y]).$$	(Theorem 4.2.1) By construction there is a surjection $\iota_M^* : A^\bullet(\Sigma_{A_n}) \twoheadrightarrow A^\bullet(M)$, and we establish[2] $$A^\bullet(M) \simeq A^\bullet(\Sigma_{A_n})/\operatorname{ann}(\Delta_M).$$
The properties of $A^\bullet(Y)$ are thus determined by how $\zeta_1,\dots,\zeta_s$ "intersect" Y.	(Theorem 3.3.8) A monomial basis in the images of h_S's under ι_M^* acquires a combinatorial interpretation as "relative nested quotients" of M.
If further ζ_i is base-point-free, then the product $\zeta_i \cdot [Y]$ can be described as a hyperplane section of a map $Y \to \mathbb{P}^m$.	(Theorem 3.2.3) The cap product $h_S \cap \Delta_M$ can be described as a principal matroid truncation of M, an operation in matroid theory that models intersecting a hyperplane with a linear subspace.

Table 1.2: Combinatorial counterparts of geometric features in Phenomenon I

Features in Phenomenon II	Their counterparts for a matroid M on $\{0,\dots,n\}$ of rank $d+1$
The polynomial $\operatorname{vol}(\underline{t})$ is positive on the positive orthant $\mathbb{R}^s_{>0}$. In particular, every term has nonnegative coefficients.	(Theorem 5.2.4) Let $\{S_1,\dots,S_d\}$ be a multiset of nonempty subsets of $\{0,\dots,n\}$, then $$\int_M h_{S_1}(M)\cdots h_{S_d}(M) = 1 \text{ or } 0,$$ and the value is 1 if $\operatorname{rk}_M(\bigcup_{j\in J} S_j) \geq \lvert J\rvert + 1$ for every $\emptyset \subsetneq J \subseteq \{1,\dots,d\}$, and 0 otherwise.
The polynomial $\operatorname{vol}(\underline{t})$ is log-concave on the positive orthant.	(Theorem 5.3.1) The polynomial $VP_M^\nabla(\underline{t})$ is Lorentzian in the sense of [BH19], and hence log-concave on its positive orthant.

Table 1.3: Combinatorial counterparts of features in Phenomenon II

[2]Such isomorphism fails in general for a surjection of Chow cohomology rings of fans $A^\bullet(\Sigma) \twoheadrightarrow A^\bullet(\Sigma')$. See [MS15, Example 6.7.13] for an example.

Chapter 2

Tropical intersection theory

In this chapter, we set up the language of tropical intersection theory, and review relevant background materials on Chow rings of matroids. While these notions originate in toric geometry, whose connection to tropical geometry is given in [FS97], familiarity with toric or tropical geometry can be helpful but not necessary. As references we point to [Ful93] and [CLS11] for toric geometry, and to [FS97], [MS15, Chapter 6], and [AHK18, §4–§5] for tropical geometry.

In §2.1, we describe Chow cohomology rings and Minkowski weights of fans, and in §2.2, we illustrate these notions in the setting of matroids. These first two sections are purely combinatorial. In §2.3, we provide the underlying geometric picture that motivates many of the combinatorial constructions, but this section is not logically necessary for future chapters except the last Chapter 7.

2.1 Chow cohomology rings and Minkowski weights

We give a brief account of Chow cohomology rings and Minkowski weights of smooth fans, which are combinatorial analogues of cohomology rings and homology classes of algebraic varieties[1].

We set the following notations and definitions for rational fans over a lattice.

- Let N be a lattice of rank n, and $N^\vee$ the dual lattice. We write $N_\mathbb{R} := N \otimes_\mathbb{Z} \mathbb{R}$.
- For $\Sigma \subset N_\mathbb{R}$ a rational fan, let $\Sigma(k)$ be the set of k-dimensional cones of Σ.

[1]We use real coefficients for Chow cohomology rings and Minkowski weights, although Chow rings of algebraic varieties initially take intrgral coefficients. The algebraic varieties that motivate the constructions here—smooth complete toric varieties and wonderful compactifications—share the feature that the Chow ring, the integral cohomology ring, and the ring of algebraic cycles modulo numerical equivalence all coincide [EH16, Appendix C.3.4]. Hence, not much is lost by tensoring with $\mathbb{R}$. In this paper, while most of our arguments work over $\mathbb{Z}$, we will always work over $\mathbb{R}$ for convenience.

- For a ray $\rho \in \Sigma(1)$, write $u_\rho \in N$ for the **primitive ray vector** that generates $\rho \cap N$.
- A fan Σ is **smooth** if, for all cones σ of Σ, the set of primitive ray vectors of σ can be extended to a basis of N. A smooth fan is **simplicial** in that every k-dimensional cone is generated by k rays.
- A fan Σ is said to be **complete** if its support $|\Sigma|$ is equal to $N_{\mathbb{R}}$.

Convention. Throughout this section, we assume that $\Sigma \subset N_{\mathbb{R}}$ is a smooth fan of dimension d, not necessarily complete.

Definition 2.1.1. The **Chow cohomology ring** $A^\bullet(\Sigma)$ of Σ is a graded $\mathbb{R}$-algebra

$$A^\bullet(\Sigma) := \frac{\mathbb{R}[x_\rho : \rho \in \Sigma(1)]}{\left\langle \prod_{\rho \in S} x_\rho \;\middle|\; S \subseteq \Sigma(1) \text{ do not form a cone in } \Sigma \right\rangle + \left\langle \sum_\rho m(u_\rho) x_\rho \;\middle|\; m \in N^\vee \right\rangle}.$$

Geometrically, the ring $A^\bullet(\Sigma)$ is the Chow ring $A^\bullet(X_\Sigma)$ of the toric variety X_Σ associated to the fan Σ. See [Dan78, §10.1] for the case where Σ is complete, and [BDP90] or [Bri96] for the general case. From this geometric description of $A^\bullet(\Sigma)$, or directly from the algebraic definition above, one can check that $A^\ell(\Sigma) = 0$ unless $0 \le \ell \le d$.

We call a linear combination of the variables x_ρ a **divisor** on Σ because geometrically it corresponds to torus-invariant divisors of X_Σ. Divisors of special interest in algebraic geometry are nef and ample divisors. They have the following combinatorial description for a complete fan Σ (equivalently, a complete toric variety X_Σ).

A divisor $D = \sum_{\rho \in \Sigma(1)} c_\rho x_\rho$ on a complete fan Σ defines a piecewise-linear function $\varphi_D : N_{\mathbb{R}} \to \mathbb{R}$, determined by being linear on each cone of Σ with $\varphi_D(u_\rho) = c_\rho$. We say that D is a **nef divisor** if φ_D is a convex function on $N_{\mathbb{R}}$, that is, $\varphi_D(u) + \varphi_D(u') \ge \varphi(u + u')$ for all $u, u' \in N_{\mathbb{R}}$. If further the inequalities $\varphi_D(u) + \varphi_D(u') \ge \varphi(u + u')$ are strict whenever u and u' are not in a common cone of Σ, we say that D is **ample**. Nef (resp. ample) divisors on Σ correspond to polytopes in $N^\vee_{\mathbb{R}}$ whose outer normal fans coarsen (resp. equal) Σ.

Theorem 2.1.2. [CLS11, Theorems 6.1.5–6.1.7] Let Σ be a smooth complete fan. A nef divisor $D = \sum_{\rho \in \Sigma(1)} c_\rho x_\rho$ on Σ defines a polytope $P_D \subset N^\vee_{\mathbb{R}}$ by

$$P_D := \{ m \in N^\vee_{\mathbb{R}} \mid m(u_\rho) \le c_\rho \ \forall \rho \in \Sigma(1) \},$$

whose outer normal fan coarsens Σ. Conversely, such polytope $P \subset N^\vee_{\mathbb{R}}$ defines a nef divisor

$$D_P := \sum_{\rho \in \Sigma(1)} \max\{ m(u_\rho) \mid m \in P \} x_\rho.$$

A nef divisor D is ample if the outer normal fan of P_D is equal to Σ.

A divisor D defines an element $[D] \in A^1(\Sigma)$, which we call the **divisor class** (of D) on Σ. We say that a divisor class $\zeta \in A^1(\Sigma)$ is **nef** (resp. **ample**) if any divisor D such that $[D] = \zeta$ is nef (resp. ample). This is well-defined because two divisors D and D' define the same divisor class if and only if $\varphi_D - \varphi_{D'}$ is a linear function on $N_{\mathbb{R}}$. In terms of polytopes, two nef divisors D and D' define the same divisor class in if and only if P_D and $P_{D'}$ are parallel translates.

Remark 2.1.3. We note that any nef divisor class $[D] \in A^1(\Sigma)$ is **effective**; that is, it can be written as non-negative linear combination $D = \sum_{\rho \in \Sigma(1)} c_\rho x_\rho$ (with $c_\rho \geq 0 \ \forall \rho \in \Sigma(1)$). This is an immediate consequence of Theorem 2.1.2: Given a nef divisor D, translating if necessary one can assume that the polytope P_D contains the origin in its relative interior.

With $A^\bullet(\Sigma)$ as an analogue of a cohomology ring, we now describe an analogue of a homology group.

Definition 2.1.4. An ℓ-dimensional **Minkowski weight** on Σ is a function $\Delta : \Sigma(\ell) \to \mathbb{R}$ such that for each $\tau \in \Sigma(\ell - 1)$, the function Δ satisfies the balancing condition

$$\sum_{\tau \prec \sigma} \Delta(\sigma) u_{\sigma \setminus \tau} \in \operatorname{span}_{\mathbb{R}}(\tau),$$

where $\sigma \setminus \tau$ denotes the unique ray of an ℓ-dimensional cone σ that is not in τ. The **support** of Δ, denoted $|\Delta|$, is the union of cones $\sigma \in \Sigma(\ell)$ such that $\Delta(\sigma) \neq 0$. We write $\mathrm{MW}_\ell(\Sigma)$ for the group (under addition) of ℓ-dimensional Minkowski weights on Σ.

The groups of Minkowski weights are analogues of homology groups because they are dual to the Chow cohomology ring in the following way.

Lemma 2.1.5. [MS15, Theorem 6.7.5][2] For $0 \leq \ell \leq d$, we have an isomorphism

$$t_\Sigma : \mathrm{MW}_\ell(\Sigma) \xrightarrow{\sim} \operatorname{Hom}(A^\ell(\Sigma), \mathbb{Z}), \quad \text{determined by } \Delta \mapsto \Big((\textstyle\prod_{\rho \in \sigma(1)} x_\rho) \mapsto \Delta(\sigma) \Big).$$

This isomorphism is an analogue of the Kronecker duality map in algebraic topology. We use it to define combinatorial analogues of some standard operations in algebraic topology.

Let us define the **cap product** by

$$A^k(\Sigma) \times \mathrm{MW}_\ell(\Sigma) \to \mathrm{MW}_{\ell-k}(\Sigma), \quad (\xi, \Delta) \mapsto \xi \cap \Delta := \Big(\sigma \mapsto (t_\Sigma \Delta)(\xi \cdot \textstyle\prod_{\rho \in \sigma(1)} x_\rho) \Big),$$

which makes $\mathrm{MW}_\bullet(\Sigma)$ a graded $A^\bullet(\Sigma)$-module. When Σ satisfies $\mathrm{MW}_d(\Sigma) \simeq \mathbb{R}$, the **fundamental class** Δ_Σ is defined as its generator (unique up to scaling), and the cap product with the fundamental class defines the map

$$\delta_\Sigma : A^\bullet(\Sigma) \to \mathrm{MW}_{d-\bullet}(\Sigma), \quad \xi \mapsto \xi \cap \Delta_\Sigma.$$

[2]Currently [MS15, Theorem 6.7.5] has a typo—it is missing $\operatorname{Hom}(\cdot, \mathbb{Z})$. The statement here was made implicitly in [Ful+95], and follows the notation of [AHK18, Proposition 5.6].

In particular, noting that $\mathrm{MW}_0(\Sigma) = \mathbb{R}$, the **degree map** is defined as

$$\int_\Sigma : A^d(\Sigma) \to \mathbb{R}, \quad \xi \mapsto \xi \cap \Delta_\Sigma.$$

If Σ is complete, one can check that $\mathrm{MW}_n(\Sigma) \simeq \mathbb{R}$, where the fundamental class Δ_Σ is $\Delta_\Sigma(\sigma) = 1 \ \forall \sigma \in \Sigma(n)$. In this case, we have the following analogue of the Poincaré duality theorem in algebraic topology.

Theorem 2.1.6. [FS97, Theorem 3.1, Proposition 4.1.(b), Theorem 4.2] For Σ a smooth complete fan, the cap product with the fundamental class Δ_Σ

$$\delta_\Sigma : A^k(\Sigma) \xrightarrow{\sim} \mathrm{MW}_{n-k}(\Sigma), \quad \xi \mapsto \xi \cap \Delta_\Sigma$$

is an isomorphism for each $0 \leq k \leq n$. Equivalently (by Lemma 2.1.5), the pairing

$$A^k(\Sigma) \times A^{n-k}(\Sigma) \to \mathbb{R}, \quad (\alpha, \beta) \mapsto \int_\Sigma \alpha\beta$$

is non-degenerate for each $0 \leq k \leq n$.

These isomorphisms make $\mathrm{MW}_\bullet(\Sigma) = \bigoplus_{i=0}^n \mathrm{MW}_i(\Sigma)$ into a graded ring when Σ is complete. We write $\mathrm{MW}^\bullet(\Sigma) := \mathrm{MW}_{n-\bullet}(\Sigma)$ for this graded ring. The resulting multiplication structure on the Minkowski weights is known as the **stable intersection**, denoted[3] $\cap_{st}$, in tropical geometry. We will only need a special case of stable intersections, which we provide explicitly in the context of matroids in §3.1 (Proposition 3.1.8).

We will need the following explicit description of the map $\delta_\Sigma : A^1(\Sigma) \xrightarrow{\sim} \mathrm{MW}_{n-1}(\Sigma)$ for nef divisor classes on a complete fan Σ. It is familiar to tropical geometers as the description of tropical hypersurfaces [MS15, Proposition 3.3.2 & Theorem 6.7.7].

Proposition 2.1.7. Let D be a nef divisor on Σ, and P_D the corresponding polytope whose outer normal fan Σ_{P_D} coarsens Σ. Then the Minkowski weight $\Delta_{P_D} := \delta_\Sigma([D]) \in \mathrm{MW}_{n-1}(\Sigma)$ given by Theorem 2.1.6 is defined by

$$\Delta_{P_D}(\tau) = \begin{cases} \ell(P_D(\sigma)) & \text{if } \exists\, \sigma \in \Sigma_{P_D}(n-1) \text{ with } |\tau| \subseteq |\sigma| \\ 0 & \text{otherwise} \end{cases} \qquad \text{for each } \tau \in \Sigma(n-1),$$

where $P_D(\sigma)$ is the edge of P_D corresponding to the cone $\sigma \in \Sigma_Q(n-1)$, and $\ell(P_D(\sigma))$ is its lattice length, i.e. the number of lattice points on $P_D(\sigma)$ minus one.

We end this subsection by noting the functoriality of the constructions here. An inclusion of fans $\iota : \Sigma' \hookrightarrow \Sigma$ defines the **pullback map** ι^*, which is a surjective map of graded rings

$$\iota^* : A^\bullet(\Sigma) \twoheadrightarrow A^\bullet(\Sigma'), \quad x_\rho \mapsto \begin{cases} x_\rho & \text{if } \rho \in \Sigma'(1) \\ 0 & \text{otherwise.} \end{cases}$$

[3] It may help to note the suggestiveness of the notations here—we have $\xi \cap \Delta = \delta_\Sigma(\xi) \cap_{st} \Delta$.

Comparing the presentations of $A^\bullet(\Sigma)$ and $A^\bullet(\Sigma')$, one checks easily that this map coincides with the quotient of $A^\bullet(\Sigma)$ by the ideal $\langle x_\rho \mid \rho \in \Sigma(1) \setminus \Sigma'(1)\rangle \subset A^\bullet(\Sigma)$. Dually, a Minkowski weight Δ' on Σ' is naturally a Minkowski weight on Σ. In this case we often abuse the notation and write Δ' for both Minkowski weights.

Remark 2.1.8. Unraveling the definitions, one checks that the cap product is functorial in the following sense: The pullback map $\iota^* : A^\bullet(\Sigma) \to A^\bullet(\Sigma')$ makes $\mathrm{MW}_\bullet(\Sigma')$ into a $A^\bullet(\Sigma)$-module. Explicitly, if $\xi \in A^\bullet(\Sigma)$ and $\Delta' \in \mathrm{MW}_\ell(\Sigma')$, then $\iota^*\xi \cap \Delta' = \xi \cap \Delta'$, where Δ' on the right hand side is considered as a Minkowski weight on Σ.

2.2 Bergman classes and Chow rings of matroids

We now specialize our discussion to matroids. We begin with the braid fan, on which matroids will arise as certain Minkowski weights.

First, we fix some notations. Let $E := \{0, 1, \dots, n\}$, and for a subset $S \subseteq E$ write $\mathbf{e}_S := \sum_{i\in S} \mathbf{e}_i$, where $\mathbf{e}_0, \dots, \mathbf{e}_n$ is the standard basis of $\mathbb{Z}^E$. Let N be the lattice $N = \mathbb{Z}^E/\mathbb{Z}\mathbf{e}_E$, and write u_S for the image of $\mathbf{e}_S$ in N. The dual lattice of N is $N^\vee = (\mathbb{Z}\mathbf{e}_E)^\perp = \{(y_0, \dots, y_n) \in \mathbb{Z}^E \mid \sum_{i=0}^n y_i = 0\}$.

The **braid fan** (of dimension n), denoted Σ_{A_n}, is the outer normal fan of the standard **permutohedron** (of dimension n), which is the polytope

$$\Pi_n := \mathrm{Conv}(w(0, 1, \dots, n) \in \mathbb{R}^E \mid \text{all permutations } w \text{ of } E).$$

Concretely, the braid fan Σ_{A_n} is a complete fan in $N_\mathbb{R}$ whose cones are $\mathrm{Cone}(u_{S_1}, \dots, u_{S_k}) \subset N_\mathbb{R}$, one for each chain of nonempty proper subsets $\emptyset \subsetneq S_1 \subsetneq \cdots \subsetneq S_k \subsetneq E$. In particular, the primitive rays of Σ_{A_n} are $\{u_S \mid \emptyset \subsetneq S \subsetneq E\}$. This fan is also known as the Coxeter complex of the type A root system, hence the notation Σ_{A_n}.

Matroids will arise as certain Minkowski weights on Σ_{A_n}. We assume familiarity with the basics of matroids, and refer to [Wel76; Oxl11] as general references. We fix the following notation for matroids: We write $U_{r,E}$ for the uniform matroid of rank r on E, and we set a matroid M to have

- ground set $E = \{0, 1, \dots, n\}$,
- $\mathcal{B}(M)$ the set of bases of M,
- rk_M the rank function of M, or simply rk when the matroid in question is clear,
- $\mathscr{L}_M$ the lattice of flats of M, which we also use to denote the set of flats,
- $\mathfrak{A}(M)$ the set of atoms of $\mathscr{L}_M$, that is, the flats of rank 1,
- $\mathrm{cl}_M(S)$ the closure of a subset $S \subseteq E$, that is, the smallest flat of M containing S,
- $Q(M)$ the base polytope of M, which is the polytope $\mathrm{Conv}(\mathbf{e}_B \mid B \in \mathcal{B}(M)) \subset \mathbb{R}^E$.

Matroids define elements of $\mathrm{MW}_\bullet(\Sigma_{A_n})$ in the following way. For the underlying geometry and a reason for the loopless condition for matroids here, see Theorem 2.3.1 and the discussion above it.

Proposition 2.2.1. Let M be a loopless matroid M of rank $r = d+1$.

(1) [MS15, Theorem 4.4.5] A function $\Delta_M : \Sigma_{A_n}(d) \to \mathbb{R}$ defined by

$$\Delta_M(\mathrm{Cone}(u_{S_1}, \dots, u_{S_d})) = \begin{cases} 1 & \text{if } S_1, \dots, S_d \text{ are flats of } M \\ 0 & \text{otherwise} \end{cases}$$

for each chain of nonempty proper subsets $\emptyset \subsetneq S_1 \subsetneq \cdots \subsetneq S_k \subsetneq E$ is a d-dimensional Minkowski weight on Σ_{A_n}.

(2) [AHK18, Proposition 5.2] Let Σ_M be the smooth fan structure on the support $|\Delta_M|$ inherited from Σ_{A_n}. In other words, Σ_M is a subfan of Σ_{A_n} whose cones are $\mathrm{Cone}(u_{F_1}, \dots, u_{F_k}) \subset N_{\mathbb{R}}$, one for each chain of nonempty proper flats $\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq E$ of M. Then the Bergman class Δ_M is the unique d-dimensional Minkowski weight on Σ_M (up to scaling).

Definition 2.2.2. With notations as in Proposition 2.2.1 above, we call the Minkowski weight Δ_M the **Bergman class** of M, and we call the fan Σ_M the **Bergman fan**[4] of M.

We will need the following description of supports of Bergman classes in Chapter 3.

Lemma 2.2.3. [MS15, Corollary 4.2.11] Let M be a loopless matroid, and $Q(M)$ its base polytope. The support $|\Delta_M|$ of its Bergman class is equal to the union of cones σ in the outer normal fan of $Q(M)$ satisfying the following condition: The corresponding face $Q(M)(\sigma)$ of σ is a base polytope of a loopless matroid.

The Chow ring of a matroid is defined as the Chow cohomology ring of its Bergman fan. Explicitly, we have the following.

Definition 2.2.4. The **Chow ring** of a loopless matroid M is the graded ring

$$A^\bullet(M) := A^\bullet(\Sigma_M) = \frac{\mathbb{R}[x_F : F \in \mathscr{L}_M \setminus \{\emptyset, E\}]}{\langle x_F x_{F'} \mid F, F' \text{ incomparable}\rangle + \langle \sum_{F \supseteq a} x_F - \sum_{G \supseteq b} x_G \mid a, b \in \mathfrak{A}(M)\rangle}.$$

We call linear combinations of the variables x_F **divisors** on M, and the elements of $A^1(M)$ **divisor classes** on M. The divisor class of $\sum_{F \supseteq a} x_F$ for any atom $a \in \mathfrak{A}(M)$ is called the **hyperplane class** of M, and is denoted $\alpha(M)$.

[4] We remark that some define the Bergman fan of M as a coarser smooth fan structure on the support $|\Sigma_M|$. A smooth fan structure on $|\Sigma_M|$ that coarsens Σ_M correspond to a choice of a building set on the lattice of flats $\mathscr{L}_M$ [AK06; FS05]. Here we will always take the smooth fan structure for Σ_M as a subfan of Σ_{A_n}.

Remark 2.2.5. The ring $A^\bullet(M)$ was first studied in [FY04] under a slightly different presentation, which for clarity is denoted $A^\bullet_{FY}(M)$ and is given as

$$A^\bullet_{FY}(M) := \frac{\mathbb{R}[z_F : F \in \mathscr{L}_M \setminus \{\emptyset\}]}{\langle z_F z_{F'} \mid F, F' \text{ incomparable}\rangle + \langle \sum_{F \supseteq a} z_F | \ a \in \mathfrak{A}(M)\rangle}.$$

That is, we have $x_F = z_F$ for every nonempty proper flat $F \in \mathscr{L}_M$, and $z_E = -\alpha$. As both presentataions $A^\bullet(M)$ and $A^\bullet_{FY}(M)$ are relevant for us, we will use the variable names x, z in a consistent manner; for example, in the summation $\sum_{F \supseteq F'} x_F$ it is implied that $F \subsetneq E$, whereas $\sum_{F \supseteq F'} z_F$ includes z_E as a summand.

Since $\mathrm{MW}(\Sigma_M) \simeq \mathbb{R}$ by Proposition 2.2.1.(2), with the Bergman class as the fundamental class, the Chow ring of a matroid M has the **degree map** $\int_M : A^d(\Sigma_M) \to \mathbb{R}$, defined by the cap product $\xi \mapsto \xi \cap \Delta_M$. Explicitly, it is determined by

$$\int_M x_{F_1} x_{F_2} \cdots x_{F_d} = 1 \text{ for every maximal chain } F_1 \subsetneq \cdots \subsetneq F_d \text{ in } \mathscr{L}_M \setminus \{\emptyset, E\}.$$

Note that the braid fan Σ_{A_n} is the Bergman fan of the Boolean matroid $U_{|E|,E}$, and its fundamental class $\Delta_{\Sigma_{A_n}}$ is the Bergman class of $U_{|E|,E}$. We will thus always identify $A^\bullet(\Sigma_{A_n}) = A^\bullet(U_{|E|,E})$.

We end this subsection with a discussion of nef and ample divisors on the brain fan, and the resulting analogous notions for Bergman fans. The following characterization of nef divisors on Σ_{A_n}, which is a specialization of Theorem 2.1.2, was recognized in various works [Edm70; Mur03; Pos09; AA17]; for a modern treatment and generalization to arbitrary Coxeter root systems we point to [Ard+20].

Proposition 2.2.6. The following are equivalent for a divisor $D = \sum_{\emptyset \subsetneq S \subsetneq E} c_S x_S \in A^1(\Sigma_{A_n})$.

(1) D is a nef divisor on Σ_{A_n},

(2) the function $c_{(\cdot)} : 2^E \to \mathbb{R}$ satisfies the **submodular property**

$$c_A + c_B \geq c_{A \cup B} + c_{A \cap B} \quad \text{for every } A, B \subseteq E \text{ where } c_\emptyset = c_E = 0,$$

(3) the normal fan of the polytope $P_D = \{m \in N^\vee_\mathbb{R} \mid m(u_S) \leq c_S \ \forall \emptyset \subsetneq S \subsetneq E\}$ coarsens Σ_{A_n},

(4) every edge of P_D is parallel to $\mathbf{e}_i - \mathbf{e}_j$ for some $i \neq j \in E$.

Remark 2.2.7. Often the polytope P_D is constructed in an affine translate of $N^\vee_\mathbb{R}$ in $\mathbb{R}^E$, for which the presentation $A^\bullet_{FY}(\Sigma_{A_n})$ is useful. Given a submodular function $c_{(\cdot)} : 2^E \to \mathbb{Z}$ with $c_\emptyset = 0$ but c_E possibly nonzero, the **generalized permutohedron** associated to $c_{(\cdot)}$ is the polytope

$$P(c) := \{y \in (\mathbb{R}^E)^\vee \mid y(\mathbf{e}_E) = c_E \text{ and } y(\mathbf{e}_S) \leq c_S \ \forall \emptyset \subsetneq S \subsetneq [n]\}.$$

This polytope lives in the translate of $N_{\mathbb{R}}^{\vee}$ where the points have coordinate sum c_E. One translates $P(c)$ to $N_{\mathbb{R}}^{\vee}$ as follows. Fix an element $i \in E$. We have

$$P(c) - c_E \mathbf{e}_i = \{m \in N_{\mathbb{R}}^{\vee} \mid m(u_S) \leq c_S - c_E \alpha_S^{(i)} \ \forall S\} \subset N_{\mathbb{R}}^{\vee}$$

where $\alpha_S^{(i)} = 1$ if $i \in S$ and 0 otherwise. Since the divisor class of $\sum_{\emptyset \subsetneq S \subsetneq E} \alpha_S^{(i)} x_S$ is the hyperplane class α in $A^1(U_{|E|,E}) = A^1(\Sigma_{A_n})$, the nef divisor class that the polytope $P(c)$ corresponds to is

$$-c_E \alpha + \sum_{\emptyset \subsetneq S \subsetneq E} c_S x_S = \sum_{\emptyset \subsetneq S \subseteq E} c_S z_S.$$

The notion of nef and ample divisors on a matroid is inherited from the braid fan. First, note that for a loopless matroid M, the inclusion of fans $\iota_M : \Sigma_M \hookrightarrow \Sigma_{A_n}$ induces the pullback map

$$\iota_M^* : A^\bullet(\Sigma_{A_n}) \to A^\bullet(M), \text{ defined by } x_S \mapsto \begin{cases} x_S & \text{if } S \subsetneq E \text{ is a flat of } M \\ 0 & \text{otherwise.} \end{cases}$$

When we wish to clarify whether a variable x_S is an element of $A^\bullet(M)$ or $A^\bullet(\Sigma_{A_n})$, we write

$$x_S(M) := \iota_M^* x_S,$$

in which case x_S is considered as an element of $A^\bullet(\Sigma_{A_n})$ and $x_S(M)$ of $A^\bullet(M)$.

The pullback map motivates the following notions regarding divisors on M. We say that a divisor (class) on M is **combinatorially nef** if it is a pullback of a nef divisor (class) on Σ_{A_n}. A **combinatorially ample** divisor (class) is defined in a likewise manner. Explicitly, a divisor $\sum_{F \in \mathscr{L}_M \setminus \{\emptyset, E\}} c_F \cdot x_F(M)$ is combinatorially nef (resp. ample) if there exists a function $a_{(\cdot)} : 2^E \to \mathbb{Z}$ with $a_\emptyset = a_E = 0$ such that $a_F = c_F$ for all flats $F \in \mathscr{L}_M$ and

$$a_A + a_B \geq a_{A \cup B} + a_{A \cap B} \text{ for every } A, B \subseteq E$$
$$\text{(resp. with strict inequality whenever } A, B \text{ incomparable).}$$

Combinatorially nef (resp. ample) divisor classes on M are closed under nonnegative linear combinations, since nef (resp. ample) divisor classes on complete fans are in general. We thus let $\overline{\mathscr{K}}_M$ (resp. $\mathscr{K}_M$) be the cone in $A^1(M)$ of combinatorially nef (resp. ample) divisor classes on M, called the **combinatorially nef** (resp. **ample**) **cone** of M.

Remark 2.2.8. It follows from Remark 2.1.3 that a combinatorially nef divisor class $[D] \in A^1(M)$ is effective; that is, it can be written as $D = \sum_F c_F x_F$ where $c_F \geq 0$ for all $F \in \mathscr{L}_M \setminus \{\emptyset, E\}$.

2.3 The geometry of matroids via wonderful compactifications

We provide the underlying algebraic geometry of the combinatorial constructions in the previous two subsections §2.1 and §2.2. While this section is not logically necessary for future chapters except Chapter 7, it may provide helpful motivation for geometrically oriented readers.

We begin by sketching how Minkowski weights arise as Chow homology classes. Let $T_N \simeq (\Bbbk^*)^n$ be the algebraic torus of a lattice N over an algebraically closed field $\Bbbk$, and let $\mathring{Y}$ be an ℓ-dimensional subvariety of T_N. An operation in tropical geometry called **tropicalization** assigns to $\mathring{Y}$ a pure ℓ-dimensional polyhedral complex in $N_{\mathbb{R}}$, denoted $\operatorname{trop}(\mathring{Y})$, with a weight function on the ℓ-dimensional cells, such that any smooth fan structure Σ' on the support of $\operatorname{trop}(\mathring{Y})$ defines a Minkowski weight on Σ'. Let us denote this Minkowski weight by $\Delta_{\Sigma'}(\operatorname{trop}(\mathring{Y}))$.

For any complete fan $\Sigma \subset N_{\mathbb{R}}$ containing Σ' as a subfan, let Y be the closure of $\mathring{Y}$ in the toric variety X_Σ, and let $[Y]$ be the class of Y in the Chow ring $A^\bullet(X_\Sigma)$. Then, under mild assumptions, the isomorphism $\delta_\Sigma : A^\bullet(X_\Sigma) \xrightarrow{\sim} \mathrm{MW}_{n-\bullet}(\Sigma)$ of Theorem 2.1.6 satisfies $\delta_\Sigma([Y]) = \Delta_{\Sigma'}(\operatorname{trop}(\mathring{Y}))$. In other words, the Minkowski weight $\Delta_{\Sigma'}(\operatorname{trop}(\mathring{Y}))$ can be considered as the Chow homology class of the variety Y in X_Σ. Furthermore, if $[Y_1]$ and $[Y_2]$ are two such Chow homology classes, then $\delta_\Sigma([Y_1]\cdot[Y_2]) = \Delta_{\Sigma'}(\operatorname{trop}(\mathring{Y}_1)) \cap_{st} \Delta_{\Sigma'}(\operatorname{trop}(\mathring{Y}_2))$, the latter being the stable intersection[5] of $\Delta_{\Sigma'}(\operatorname{trop}(\mathring{Y}_1))$ and $\Delta_{\Sigma'}(\operatorname{trop}(\mathring{Y}_2))$. See [MS15, §6.4 & §6.7] for proofs and further details.

Matroids arise in this context by setting $\mathring{Y}$ to be linear subvarieties. Let us now describe this in detail, and describe the connection to Bergman classes and Chow rings of matroids.

Let M be a loopless matroid on E of rank $r = d+1$ realizable over a field $\Bbbk$, which we may assume to be algebraically closed. A **realization** $\mathscr{R}(M)$ of M consists of any of the following equivalent pieces of data:

- a list of vectors $E = \{v_0, \ldots, v_n\}$ spanning a $\Bbbk$-vector space $V \simeq \Bbbk^r$,
- a surjection $\Bbbk^{n+1} \twoheadrightarrow V$ where $\mathbf{e}_i \mapsto v_i$, or
- an injection $\mathbb{P}V^* \hookrightarrow \mathbb{P}^n_{\Bbbk}$, dualizing the surjection $\Bbbk^{n+1} \twoheadrightarrow V$.

[5] A sketch of the definition of stable intersections of Minkowski weights, reminiscent of the moving lemma in intersection theory, is as follows. Considering Minkowski weights as weighted polyhedral complexes in $N_{\mathbb{R}}$, generically translate the Minkowski weights an $\epsilon > 0$ amount, so that all the resulting weighted cones intersect transversally, and then take the limit of these transversal intersections as $\epsilon \to 0$. See [FS97], [MS15, §3.6 & §6.7], or [JY16] for details

For a realization $\mathscr{R}(M)$ of M with $\mathbb{P}V^* \hookrightarrow \mathbb{P}^n$, the coordinate hyperplanes of $\mathbb{P}^n$ intersect with $\mathbb{P}V^*$ to give the associated **hyperplane arrangement** $\mathcal{A}_{\mathscr{R}(M)}$ on $\mathbb{P}V^*$, which is encoded by the flats of M in the following way. For each nonempty flat F of M, let L_F be a linear subspace of V^* defined by

$$L_F := \{f \in V^* \mid f(v_i) = 0 \ \forall v_i \in F\},$$

and let $\mathbb{P}L_F$ be the linear subvariety of $\mathbb{P}V^*$. The hyperplanes of $\mathcal{A}_{\mathscr{R}(M)}$ are $\{\mathbb{P}L_a\}_{a\in\mathfrak{A}(M)}$ corresponding to the atoms, and more generally, a flat F of rank c corresponds to the c-codimensional linear subvariety $\mathbb{P}L_F$.

We denote by $\mathring{Y}_{\mathscr{R}(M)}$ the hyperplane arrangement complement $\mathbb{P}V^* \setminus \bigcup \mathcal{A}_{\mathscr{R}(M)}$. It is a linear subvariety of an algebraic torus in the following way. The algebraic torus $T_N = (\mathbb{k}^*)^{n+1}/\mathbb{k}^*$ of the lattice $N = \mathbb{Z}^{n+1}/\mathbb{Z}(1,1,\dots,1)$ is the complement of the union of coordinate hyperplanes in $\mathbb{P}^n$, and hence $\mathring{Y}_{\mathscr{R}(M)}$ is the intersection of $\mathbb{P}V^*$ with T_N.

Theorem 2.3.1 below relates the linear subvariety $\mathring{Y}_{\mathscr{R}(M)} \subset T_N$ to the Bergman class of M via tropicalization. Note that if M had loops, then $\mathbb{P}V^*$ is contained in a coordinate hyperplane of $\mathbb{P}^n$, so that $\mathring{Y}_{\mathscr{R}(M)}$ is empty, hence our loopless assumption for M.

Theorem 2.3.1. [MS15, Theorem 4.1.11] Let $\mathscr{R}(M)$ be a realization of a loopless matroid M, and let $\mathring{Y}_{\mathscr{R}(M)}$ be the associated hyperplane arrangement complement. Then the support of $\operatorname{trop}(\mathring{Y}_{\mathscr{R}(M)})$ equals the support of the Bergman fan Δ_M, and hence we have

$$\Delta_{\Sigma_{A_n}}(\mathring{Y}_{\mathscr{R}(M)}) = \Delta_M.$$

In other words, the Bergman class Δ_M corresponds to the Chow homology class of the closure $Y_{\mathscr{R}(M)}$ of $\mathring{Y}_{\mathscr{R}(M)}$ in the toric variety $X_{\Sigma_{A_n}}$ of the braid fan Σ_{A_n}. The variety $Y_{\mathscr{R}(M)}$ is called the **wonderful compactification** of the hyperplane arrangement complement $\mathring{Y}_{\mathscr{R}(M)}$.

Remark 2.3.2. The wonderful compactification $Y_{\mathscr{R}(M)}$ can be described in two equivalent ways [DP95, §3.2].

(1) The variety $Y_{\mathscr{R}(M)}$ is obtained as a series of blow-ups on $\mathbb{P}V^*$ by the following process: First blow-up the points $\{\mathbb{P}L_F\}_{\operatorname{rk}(F)=\operatorname{rk}(E)-1}$, then blow-up the strict transforms of the lines $\{\mathbb{P}L_F\}_{\operatorname{rk}(F)=\operatorname{rk}(E)-2}$, and continue until having blown-up strict transforms of $\{\mathbb{P}L_F\}_{\operatorname{rk}(F)=1}$. We denote by $\pi_{\mathscr{R}(M)} : Y_{\mathscr{R}(M)} \to \mathbb{P}V^*$ be the blow-down map.

(2) For each nonempty flat F of M, the projection away from the linear subvariety $\mathbb{P}L_F \subset \mathbb{P}V^*$ is a rational map $\mathbb{P}V^* \dashrightarrow \mathbb{P}(V^*/L_F)$. The variety $Y_{\mathscr{R}(M)}$ then is the (closure of) the graph of the rational map

$$\mathbb{P}V^* \dashrightarrow \prod_{F\in\mathscr{L}_M\setminus\{\emptyset\}} \mathbb{P}(V^*/L_F).$$

When $U_{n+1,n+1}$ is realized as the standard basis of $\Bbbk^{n+1}$, the associated wonderful compactification is the toric variety $X_{\Sigma_{A_n}}$ of the braid fan. It is obtained from $\mathbb{P}^n$ by blowing up the coordinate points, then the (strict transforms of) coordinate lines, and so forth. Let us write $\pi_{A_n} : X_{\Sigma_{A_n}} \to \mathbb{P}^n$ for the blow-down map. Then for a realization $\mathbb{P}V^* \hookrightarrow \mathbb{P}^n$ of a loopless matroid M, Remark 2.3.2.(1) above expresses the wonderful compactification $Y_{\mathscr{R}(M)}$ as the strict transform of $\mathbb{P}V^* \subset \mathbb{P}^n$ under the sequence of blow-ups π_{A_n}. In other words, we have a diagram

$$\begin{tikzcd} Y_{\mathscr{R}(M)} \arrow[r, hook, "\iota_M"] \arrow[d, "\pi_{\mathscr{R}(M)}"'] & X_{\Sigma_{A_n}} \arrow[d, "\pi_{A_n}"] \\ \mathbb{P}V^* \arrow[r, hook] & \mathbb{P}^n. \end{tikzcd}$$

The boundary of $Y_{\mathscr{R}(M)} \setminus \mathring{Y}_{\mathscr{R}(M)}$ consists of the exceptional divisors $\mathcal{E}_F$ obtained by blowing up (strict transforms of) $\mathbb{P}L_F$. These divisors have simple-normal-crossings [DP95], and consequently the intersection theory of the boundary divisors of $Y_{\mathscr{R}(M)}$ is encoded in the matroid. More precisely, the Chow ring $A^\bullet(Y_{\mathscr{R}(M)})$ of the variety $Y_{\mathscr{R}(M)}$ is isomorphic to the Chow cohomology ring $A^\bullet(M)$ of the Bergman fan of M [FY04, Corollary 2].

Remark 2.3.3. We note the following geometric observations about the presentation

$$A^\bullet(Y_{\mathscr{R}(M)}) \simeq A^\bullet(M) = \frac{\mathbb{R}[x_F : F \in \mathscr{L}_M \setminus \{\emptyset, E\}]}{\langle x_F x_{F'} \mid F, F' \text{ incomparable}\rangle + \langle \sum_{F \supseteq a} x_F - \sum_{G \supseteq b} x_G \mid a, b \in \mathfrak{A}(M)\rangle}.$$

(1) The variables x_F correspond to the exceptional divisors $\mathcal{E}_F$ obtained by blowing up (strict transforms of) $\mathbb{P}L_F$.

(2) The quadric relations $x_F x_{F'} = 0$ reflect that two exceptional divisors from blowing up two non-intersecting linear subspaces do not intersect.

(3) The linear relations defining $A^\bullet(M)$ reflect that for any atom $a \in \mathfrak{A}(M)$, we have $-z_E = \alpha(M) = \sum_{F \supseteq a} x_F = \pi^*_{\mathscr{R}(M)} h$ where $h = c_1(\mathscr{O}_{\mathbb{P}V^*}(1))$ is the hyperplane class of $\mathbb{P}V^*$.

(4) Under $A^\bullet(Y_{\mathscr{R}(M)}) \simeq A^\bullet(M)$ and $A^\bullet(X_{\Sigma_{A_n}}) \simeq A^\bullet(\Sigma_{A_n})$, the pullback map $\iota_M^* : A^\bullet(X_{\Sigma_{A_n}}) \to A^\bullet(Y_{\mathscr{R}(M)})$ along the closed embedding $\iota_M : Y_{\mathscr{R}(M)} \hookrightarrow X_{\Sigma_{A_n}}$ is the pullback map of Chow cohomology rings of Σ_M and Σ_{A_n} induced by the inclusion of fans $\Sigma_M \hookrightarrow \Sigma_{A_n}$.

(5) A divisor class $D \in A^1(M)$ is an combinatorially ample (nef) if and only if there exists an ample (nef) divisor class L on X_{A_n} such that $\iota_M^* L = D$. Combinatorially ample (nef) divisors are ample (nef) on the variety $Y_{\mathscr{R}(M)}$.

Remark 2.3.4 (Relation to Phenomenon I)**.** The toric variety $X_{\Sigma_{A_n}}$ is also called the permutohedral variety (of dimension n). The geometry and the combinatorics of the permutohedral variety have been widely studied in various contexts including moduli spaces [LM00; BB11], convex optimization [Edm70; Mur03], Hopf monoids [DF10; AA17], and lattice polyhedra [PRW08; Pos09].

In our case, the variety $X_{\Sigma_{A_n}}$ plays the role of "the variety whose Chow ring has been well-studied" in Phenomenon I, and the wonderful compactification $Y_{\mathscr{R}(M)}$ plays the role of "the subvariety Y whose Chow ring is an interest of study." Moreover, since we have a surjection $A^\bullet(X_{\Sigma_{A_n}}) \twoheadrightarrow A^\bullet(M) \simeq A^\bullet(Y_{\mathscr{R}(M)})$, we have an isomorphism $A^\bullet(Y_{\mathscr{R}(M)}) \simeq A^\bullet(X_{\Sigma_{A_n}}) / \operatorname{ann}([Y_{\mathscr{R}(M)}])$. We will realize this isomorphism combinatorially in §4.2 as Theorem 4.2.1.

Chapter 3

The simplicial presentation and its monomials

We now introduce the main object of study: a new presentation of the Chow ring of a matroid which we call the simplicial presentation $A^\bullet_\nabla(M)$ of $A^\bullet(M)$. While algebraically this involves only an upper triangular linear change of variables, its geometric and combinatorial implications are far-reaching as we will see in subsequent chapters.

After a combinatorial preparation in §3.1, we introduce the simplicial presentation in §3.2 and show that its variables correspond to a matroid operation called principal truncations. In §3.3, we extend this correspondence to establish a combinatorial interpretation of a monomial basis of the Chow ring of a matroid.

3.1 Matroid quotients, principal truncations, and matroid intersections

We first prepare by reviewing the relevant combinatorial notions. We point to [Oxl11, §7] and [Ham17, §2.3] for further details.

Let M and M' be matroids on a common ground set E.

Definition 3.1.1. The matroid M' is a **(matroid) quotient** of M, written $f : M' \twoheadleftarrow M$, if any every flat of M' is also a flat of M. In particular, if M and M' are loopless, then $f : M' \twoheadleftarrow M$ if and only if $\Sigma_{M'} \subseteq \Sigma_M$.

Example 3.1.2. Any matroid on ground set E is a quotient of the Boolean matroid $U_{|E|,E}$. Any Bergman fan of a loopless matroid is a subfan of the braid fan.

Example 3.1.3 (Realizable matroid quotients)**.** Matroid quotients model linear surjections (dually, linear injections) in the following way. Let M and M' have realizations by $\Bbbk^E \twoheadrightarrow$

V and $\Bbbk^E \twoheadrightarrow V'$ (respectively). Then having a commuting diagram

$$\begin{tikzcd} \Bbbk^E \arrow[r, two heads] \arrow[d, equal] & V \arrow[d, two heads] \\ \Bbbk^E \arrow[r, two heads] & V' \end{tikzcd} \quad \text{or dually,} \quad \begin{tikzcd} \mathbb{P}^E_\Bbbk \arrow[d, equal] & \mathbb{P}V^* \arrow[l, hook'] \\ \mathbb{P}^E_\Bbbk & \mathbb{P}V'^* \arrow[l, hook'] \arrow[u, hook] \end{tikzcd}$$

implies that M' is a matroid quotient of M. Matroid quotients $M' \twoheadleftarrow M$ arising in this way are called realizable matroid quotients. A matroid quotient $M' \twoheadleftarrow M$ with M' and M both realizable over a same field need not be realizable (for an example, see [BGW03, §1.7.5]).

For a matroid quotient $f : M' \twoheadleftarrow M$, the f**-nullity** of a subset $A \subseteq E$ is defined to be

$$n_f(A) := \mathrm{rk}_M(A) - \mathrm{rk}_{M'}(A).$$

We say that M' is an **elementary (matroid) quotient** of M if $n_f(E) = 1$, or equivalently if $\mathrm{rk}(M') = \mathrm{rk}(M) - 1$. An elementary quotient of M corresponds to a **modular cut** $\mathcal{K}$ of M, which is a nonempty collection of flats $\mathcal{K} \subset \mathscr{L}_M$ satisfying

1. if $F_1 \in \mathcal{K}$ and $F_1 \subset F_2$, then $F_2 \in \mathcal{K}$, and
2. if $F_1, F_2 \in \mathcal{K}$ and $\mathrm{rk}_M(F_1) + \mathrm{rk}_M(F_2) = \mathrm{rk}_M(F_1 \cup F_2) + \mathrm{rk}_M(F_1 \cap F_2)$, then $F_1 \cap F_2 \in \mathcal{K}$.

A modular cut $\mathcal{K}$ of M defines an elementary quotient $M' \twoheadleftarrow M$ by

$$\mathscr{L}_{M'} := \{F \in \mathscr{L}_M : F \text{ is not covered by an element of } \mathcal{K}\} \cup \mathcal{K}.$$

Conversely, given an elementary quotient $f : M' \twoheadleftarrow M$, one recovers the modular cut $\mathcal{K}$ of M defining the elementary quotient by

$$\mathcal{K} = \{F \in \mathscr{L}_{M'} : n_f(F) = 1\}.$$

We write $M' \overset{\mathcal{K}}{\twoheadleftarrow} M$ to denote an elementary quotient of M given by a modular cut $\mathcal{K}$.

Example 3.1.4. Let M have a realization $\Bbbk^E \twoheadrightarrow V$. For $\mathcal{K}$ a modular cut of M, let $v_\mathcal{K}$ be a general vector[1] in $\bigcap_{F \in \mathcal{K}} \mathrm{span}_\Bbbk(F)$. Dually, with the notation as in §2.3, we have a general hyperplane $H_\mathcal{K} = \{f \in V^* \mid f(v_\mathcal{K}) = 0\}$ in V^* containing $\bigcup_{F \in \mathcal{K}} L_F$. Let us consider the commuting diagram

$$\begin{tikzcd} \Bbbk^E \arrow[r, two heads] \arrow[d, equal] & V \arrow[d, two heads] \\ \Bbbk^E \arrow[r, two heads] & V / \mathrm{span}_\Bbbk(v_\mathcal{K}) \end{tikzcd} \quad \text{or dually,} \quad \begin{tikzcd} \mathbb{P}^E_\Bbbk \arrow[d, equal] & \mathbb{P}V^* \arrow[l, hook'] \\ \mathbb{P}^E_\Bbbk & \mathbb{P}H_\mathcal{K}. \arrow[l, hook'] \arrow[u, hook] \end{tikzcd}$$

[1]For $v_\mathcal{K}$ to be sufficiently general and nonzero, the field $\Bbbk$ must be large enough (let us assume infinite), and the elementary matroid quotient defined by the modular cut $\mathcal{K}$ must be realizable.

The map $\Bbbk^E \twoheadrightarrow V / \operatorname{span}_{\Bbbk}(v_{\mathcal{K}})$ is a realization of the matroid M' of the elementary quotient $M' \overset{\mathcal{K}}{\twoheadleftarrow} M$ defined by $\mathcal{K}$. Dually, with the notation as in §2.3, the associated hyperplane arrangement $\mathcal{A}_{\mathscr{R}(M')}$ is the intersection of $\mathbb{P}H_{\mathcal{K}}$ with the coordinate hyperplanes in $\mathbb{P}^E_{\Bbbk}$. Equivalently, the hyperplane arrangement $\mathcal{A}_{\mathscr{R}(M')}$ is the intersection of $\mathbb{P}H_{\mathcal{K}}$ with the hyperplanes in the hyperplane arrangement $\mathcal{A}_{\mathscr{R}(M)}$ under the inclusion $\mathbb{P}H_{\mathcal{K}} \subset \mathbb{P}V^*$.

Of particular interest in our case is when $\mathcal{K}$ is the interval $[F, E] \subset \mathscr{L}_M$, since an interval in $\mathscr{L}_M$ is always a modular cut. We call the resulting elementary quotient, denoted $T_F(M)$, the **principal truncation** of M associated to the flat F. We note an explicit description of principal truncations.

Proposition 3.1.5. [Oxl11, Exercise 7.2.4.] The principal truncation $T_F(M)$ of a matroid M associated to a nonempty flat $F \in \mathscr{L}_M$ has bases

$$\mathcal{B}(T_F(M)) = \{B \setminus f \text{ such that } B \in \mathcal{B}(M) \text{ and } f \in B \cap F \neq \emptyset\},$$

and the flats of $T_F(M)$ partition into two sets $\mathscr{L}_{T_F(M)} = \mathcal{K} \sqcup \mathcal{L}$ according to their f-nullities by

$$\begin{aligned}
\mathcal{K} &= \{G \in \mathscr{L}_{T_F(M)} \mid n_f(G) = 1\} = \{G \in \mathscr{L}_M \mid F \subseteq G\}, \\
\mathcal{L} &= \{G \in \mathscr{L}_{T_F(M)} \mid n_f(G) = 0\} = \{G \in \mathscr{L}_M \mid G \text{ not covered by an element in } [F, E]\}.
\end{aligned}$$

Remark 3.1.6. In Example 3.1.4, if $\mathcal{K} = [F, E]$ for some flat F, then we can set $v_{\mathcal{K}} = v_F$, a general vector in $\operatorname{span}_{\Bbbk}(F)$, and dually, we can set $H_{\mathcal{K}} = H_F$, a general hyperplane in V^* containing L_F.

We end our combinatorial preparation by illustrating the relevance of matroid quotients to Minkowski weights on braid fans via the notion of matroid intersections.

Definition 3.1.7. The **matroid intersection** of two matroids M and N on a common ground set E is a new matroid $M \wedge N$ on E whose spanning sets $\mathcal{S}(M \wedge N)$ are $\{S \cap S' \mid S \in \mathcal{S}(M),\ S' \in \mathcal{S}(N)\}$.

The matroid $M \wedge N$ is a matroid quotient of both M and N. Matroid intersection behaves well in relation to Minkowski weights in the following way. Recall that the isomorphism $A^\bullet(\Sigma_{A_n}) \simeq \mathrm{MW}_{n-\bullet}(\Sigma_{A_n})$ of Theorem 2.1.6 makes $\mathrm{MW}_{n-\bullet}(\Sigma_{A_n})$ into a graded ring, with multiplication called the stable intersection $\cap_{st}$. The following proposition states that stable intersections of Bergman classes are Bergman classes of matroid intersections.

Proposition 3.1.8. [Spe08, Theorem 4.11], [Ham17, Remark 2.31] Let M and N be two matroids on a common ground set E, and let Δ_M and Δ_N be their Bergman classes, which are Minkowski weights on Σ_{A_n}. Then we have

$$\Delta_M \cap_{st} \Delta_N = \begin{cases} \Delta_{M \wedge N} & \text{if } M \wedge N \text{ is loopless} \\ 0 & \text{otherwise.} \end{cases}$$

3.2 The variables of the simplicial presentation

We now define a new presentation $A^\bullet_\nabla(M)$ of the Chow ring of a matroid M, and discuss its first properties. The key result here is that the variables of $A^\bullet_\nabla(M)$ correspond to principal truncations of M.

We prepare by noting a distinguished set of nef divisors on Σ_{A_n} and their polytopes considered in [Pos09]. For a nonempty subset S of E, denote by

$$\nabla_S := \mathrm{Conv}(-\mathbf{e}_i \mid i \in S) \subset \mathbb{R}^E$$

the negative standard simplex of S. As the edges of ∇_S are parallel translates of $\mathbf{e}_i - \mathbf{e}_j$ for $i \neq j \in S$, Proposition 2.2.6 (in the form of Remark 2.2.7) implies that ∇_S is a polytope with the corresponding nef divisor

$$h_S := -\sum_{S \subseteq T} z_T \in A^\bullet_{FY}(\Sigma_{A_n}).$$

These divisors were considered in [Pos09][2] and implicitly in [Ham17]. We now consider the presentation of $A^\bullet(M)$ given by pullbacks of these nef divisors of (negative) standard simplices. For M a loopless matroid on E, and $\emptyset \neq S \subseteq E$, denote $h_S(M) := \iota_M^* h_S$. If $F = \mathrm{cl}_M(S)$ is the smallest flat containing S, note that we have

$$h_S(M) := \iota_M^* h_S = -\sum_{S \subseteq T} z_T(M) = -\sum_{F \subseteq G \in \mathscr{L}_M} z_G(M) = \iota_M^* h_F, \tag{3.1}$$

as $z_T(M) = \iota_M^* z_T = 0$ for all $T \subseteq E$ not a flat of M. By construction, the elements $h_F(M) \in A^1(M)$ are (combinatorially) nef divisor classes on M. We will simply write h_F for $h_F(M)$ when there is no confusion.

Definition 3.2.1. For M a loopless matroid on E, the **simplicial presentation** $A^\bullet_\nabla(M)$ of the Chow ring of M is the presentation of $A^\bullet(M)$ whose generators are $\{h_F\}_{F \in \mathscr{L}_M \setminus \{\emptyset\}}$ where

$$h_F := -\sum_{F \subseteq G} z_G \in A^\bullet_{FY}(M).$$

The variable h here stands for "hyperplane"; for the geometric origin of the simplicial presentation see Remark 3.2.6. The linear change of variables from $\{z_F\}_{F \in \mathscr{L}_M \setminus \{\emptyset\}}$ to $\{h_F\}_{F \in \mathscr{L}_M \setminus \{\emptyset\}}$ is evidently invertible, given by an upper triangular matrix. Explicitly, by Möbius inversion we have

$$-z_F = \sum_{F \subseteq G} \mu(F, G) h_G \tag{3.2}$$

[2] We note a minor difference that in [Pos09] the author uses y_S to denote the nef divisor of the standard simplex of S instead of the negative standard simplex.

where μ is the Möbius function on the lattice $\mathcal{L}_M$. Thus, the explicit presentation of $A^\bullet_\nabla(M)$ is

$$A^\bullet_\nabla(M) := \mathbb{R}[h_F : F \in \mathscr{L}_M \setminus \{\emptyset\}] \,/\, (I + J)$$

where

$$I = \langle h_a : a \in \mathfrak{A}(M)\rangle \text{ and}$$

$$J = \left\langle \left(\sum_{F \subseteq G} \mu(F,G) h_G\right)\left(\sum_{F' \subseteq G'} \mu(F',G') h_{G'}\right) : F, F' \text{ incomparable} \right\rangle.$$

Denote by $\mathscr{L}_M^{\geq 2}$ the set of flats of M of rank at least 2. Noting that $h_a = 0 \in A^\bullet_\nabla(M)$ for any atom $a \in \mathfrak{A}(M)$, we define $\{h_F \mid F \in \mathscr{L}_M^{\geq 2}\}$ to be the **simplicial generators** of the Chow ring of M. They form a basis of $A^1_\nabla(M)$.

Remark 3.2.2. When the matroid M is the cyclic matroid of the complete graph K_{n-1} on $n-1$ vertices, the Chow ring of M is the cohomology ring of the Deligne-Knudson-Mumford space $\overline{\mathcal{M}}_{0,n}$ of rational curves with n marked points [DP95, §4.3], [MS15, Theorem 6.4.12]. In this case, after suitable modifications[3] the simplicial presentation recovers the Etingof-Henriques-Kamnitzer-Rains-Singh presentation of the cohomology ring of $\overline{\mathcal{M}}_{0,n}$ [Eti+10; Sin04]. In this presentation, the author of [Dot19] showed that the cohomology ring of $\overline{\mathcal{M}}_{0,n}$ is Koszul because it has a quadratic Gröbner basis. In the classical presentation, the Chow ring of any matroid with rank > 3 has no quadratic Gröbner basis.

The following theorem, which relates the variables of the simplicial presentation to principal truncations, is the key property of the simplicial presentation that we use throughout this paper.

Theorem 3.2.3. Let M be a loopless matroid on E, and S a nonempty subset of E. Denote by H_S be the matroid whoses bases are $B(H_S) = \{E \setminus i : i \in E\}$, and write F for the smallest flat of M containing S. Then $H_S \wedge M = T_F(M)$, and the nef divisor class $h_S \in A^1_\nabla(\Sigma_{A_n})$ satisfies

$$h_S \cap \Delta_M = \begin{cases} \Delta_{T_F(M)} & \text{if } \mathrm{rk}_M(S) > 1 \\ 0 & \text{otherwise.} \end{cases}$$

The theorem will follow mostly from the following lemma.

Lemma 3.2.4. Let M be a loopless matroid on E, and S a nonempty subset of E. Denote by H_S be the matroid whoses bases are $B(H_S) = \{E \setminus i : i \in E\}$, and write F for the smallest flat of M containing S. Then we have

$$H_S \wedge M = T_F(M),$$

[3] One uses the minimal building set instead of the maximal building set.

and hence we have

$$\Delta_{H_F} \cap_{st} \Delta_M = \begin{cases} \Delta_{T_F(M)} & \text{if } \mathrm{rk}_M(F) > 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. A matroid is loopless if and only if the emptyset $\emptyset$ is a flat, and by Proposition 3.1.5, the matroid $T_F(M)$ is thus loopless if and only if $\emptyset$ is not covered by an element in the interval $[F, E]$. This happens if and only if $\mathrm{rk}_M(F) > 1$, and hence the second statement follows from the first by Proposition 3.1.8.

By definition of $H_S \wedge M$, the minimal elements in the set of spanning sets $\mathcal{S}(H_F \wedge M)$ are $B \setminus i$ where $B \in \mathcal{B}(M)$ and $i \in B \cap S \neq \emptyset$. Since minimal spanning sets are bases, we have

$$\mathcal{B}(H_S \wedge M) = \{B \setminus i \text{ such that } B \in \mathcal{B}(M),\ i \in B \cap S \neq \emptyset\}.$$

When $S = F$, this is the description of the bases of $T_F(M)$ in Proposition 3.1.5, so it remains to show $H_S \wedge M = H_F \wedge M$. Evidently, we have $\mathcal{B}(H_S \wedge M) \subseteq \mathcal{B}(H_F \wedge M)$ since $S \subseteq F$. For the other inclusion, suppose we have a basis $B \setminus f$ of $H_F \wedge M$ where $B \in \mathcal{B}(M)$ and $f \in B \cap F \neq \emptyset$. Note that $\mathrm{rk}_M((B \setminus f) \cup S) = \mathrm{rk}_M((B \setminus f) \cup F) = \mathrm{rk}_M(E)$, where the first equality follows from F being the closure of S. Since $B \setminus f$ is independent in M, we thus conclude that there exists an element $s \in S$ such that $(B \setminus f) \cup s$ is a basis of M, so that $B \setminus f$ is a basis of $H_S \wedge M$ also. □

Proof of Theorem 3.2.3. Let $\delta_{\Sigma_{A_n}} : A^\bullet(\Sigma_{A_n}) \xrightarrow{\sim} \mathrm{MW}_{n-\bullet}(\Sigma_{A_n})$ be the isomorphism map in Theorem 2.1.6. We claim that $\delta_{\Sigma_{A_n}}(h_S) = \Delta_{H_S}$, which is proved in Lemma 3.2.5 below. Our desired statement then follows immediately from Lemma 3.2.4, since $h_S \cap \Delta_M = (\delta_{\Sigma_{A_n}}(h_S)) \cap_{st} \Delta_M$ by definition of stable intersection $\cap_{st}$. □

Lemma 3.2.5. Let $\delta_{\Sigma_{A_n}} : A^\bullet(\Sigma_{A_n}) \xrightarrow{\sim} \mathrm{MW}_{n-\bullet}(\Sigma_{A_n})$ be the isomorphism map in Theorem 2.1.6. Then we have

$$\delta_{\Sigma_{A_n}}(h_S) = \Delta_{H_S}$$

Proof. We claim that the support $|\Delta_{H_S}|$ of Δ_{H_S} is equal to the support of the $(n-1)$-skeleton of the outer normal fan of negative standard simplex ∇_S. If this is the case, then Proposition 2.1.7 implies $\delta_{\Sigma_{A_n}}(h_S) = \Delta_{H_S}$ because all the edges of the negative standard simplex ∇_S has lattice length 1.

Now, for the claim, note first that the translate $\nabla_S + \mathbf{e}_E$ of ∇_S is $\mathrm{Conv}(\mathbf{e}_{E \setminus i} \mid i \in S) \subset \mathbb{R}^E$, which is equal to the base polytope $Q(H_S)$ of H_S. Since every face of $Q(H_S)$, except for the vertices, are base polytopes of loopless matroids, by Lemma 2.2.3 the support of $|\Delta_{H_S}|$ equals the support of the $(n-1)$-dimensional skeleton of the outer normal fan of ∇_S. □

Theorem 3.2.3 encodes the combinatorics of the following geometric motivation for the simplicial presentation.

Remark 3.2.6. We recall the following standard fact in algebraic geometry. Let $L \subset V^*$ be an inclusion of vector spaces, so that $\mathbb{P}L$ is a linear subvariety of $\mathbb{P}V^*$. Let $X = \mathrm{Bl}_{\mathbb{P}L}\,\mathbb{P}V^*$ be the blow-up of $\mathbb{P}V^*$ along $\mathbb{P}L$, which is also the graph of the rational map $\mathbb{P}V^* \dashrightarrow \mathbb{P}(V^*/L)$, and let $\pi : X \to \mathbb{P}V^*$ be the blow-down map. Denote by $h = c_1(\mathscr{O}_{\mathbb{P}^n}(1))$ the hyperplane class, and by $\mathcal{E}$ the exceptional divisor of the blow-up. Then the map $\mathrm{Bl}_{\mathbb{P}L}\,\mathbb{P}V^* \to \mathbb{P}(V^*/L)$ corresponds to the linear series $|\mathscr{O}_X(\pi^*h - \mathcal{E})|$, so the sections of the line bundle $\mathscr{O}_X(\pi^*h - \mathcal{E})$ correspond to the hyperplanes in $\mathbb{P}V^*$ that contain $\mathbb{P}L$.

Now, suppose M has a realization $\mathscr{R}(M)$ over an algebraically closed field $\Bbbk$ as $\mathbb{P}V^* \hookrightarrow \mathbb{P}^n$, and let notations be as in §2.3. The geometry of $A^\bullet(Y_{\mathscr{R}(M)}) \simeq A^\bullet(M)$ in Remark 2.3.3 implies

$$h_F = \sum_{G \supseteq F} -z_G = -z_E - \sum_{G \supseteq F} x_G = \pi^*_{\mathscr{R}(M)} h - \sum_{G \supseteq F} \mathcal{E}_G,$$

and hence h_F represents the divisor class of the strict transform of a general hyperplane in $\mathbb{P}V^*$ containing the linear subvariety $\mathbb{P}L_F$. Thus multiplying by h_F corresponds to intersecting by a general hyperplane in $\mathbb{P}V^*$ containing $\mathbb{P}L_F$, which corresponds to the principal truncation $T_F(M)$ by Example 3.1.4 (in the form of Remark 3.1.6). More precisely, we have $h_F \cdot [Y_{\mathscr{R}(M)}] = [Y_{\mathscr{R}(T_F(M))}] \in A^\bullet(X_{\Sigma_{A_n}})$. Theorem 3.2.3 is the combinatorial generalization of this geometric observation.

Remark 3.2.7 (cf. Phenomenon I)**.** Suppose M has a realization $\mathscr{R}(M)$ by $\mathbb{P}V^* \hookrightarrow \mathbb{P}^n$. By the second description in Remark 2.3.2.(2), the wonderful compactification $Y_{\mathscr{R}(M)}$ is embedded in the product of projective spaces $\prod_{F \in \mathscr{L}_M \setminus \{\emptyset\}} \mathbb{P}(V^*/L_F)$. We described h_F as a divisor class represented by the strict transform of a general hyperplane in $\mathbb{P}V^*$ containing $\mathbb{P}L_F$ in the previous Remark 3.2.6. Alternatively, the variable h_F thus represents the base-point-free divisor obtained as the hyperplane class pullback of the map $Y_{\mathscr{R}(M)} \to \mathbb{P}(V^*/L_F)$. In particular, the divisor classes $h_S \in A^\bullet(X_{\Sigma_{A_n}})$ play the role of "distinguished base-point-free divisor classes" in Phenomenon I, and their intersection product with $[Y_{\mathscr{R}(M)}]$ can be described by their hyperplane pullbacks, which we have interpreted combinatorially as principal truncations.

Remark 3.2.8. In the classical presentation $A^\bullet(\Sigma_{A_n})$, the cap product $x_S \cap \Delta_M$ is almost never a Bergman class of a matroid—it is a Minkowski weight that usually has negative weights on some cones. This reflects the geometry that the divisor x_S is effective but usually not nef.

Notation. Let us fix a notation for the rest of the paper: For a nonempty subset $S \subseteq E$, we denote by H_S the matroid with bases

$$\mathcal{B}(H_S) := \{E \setminus i : i \in S\},$$

or equivalently, $H_S = U_{|E \setminus S|, E \setminus S} \oplus U_{|S|-1, S}$.

3.3 A monomial basis of the simplicial presentation and relative nested quotients

We introduce the notion of relative nested quotients, which are relative generalizations of (loopless) Schubert matroids in matroid theory, and we show that they are in in bijection with elements of a monomial basis of $A^\bullet_\nabla(M)$.

We start by producing a monomial basis of $A^\bullet_\nabla(M)$ via the Gröbner basis computation in [FY04]. Pick a total order $>$ on elements of $\mathscr{L}_M$ such that $F > G$ if $\mathrm{rk}_M(F) \leq \mathrm{rk}_M(G)$, and take the induced lex monomial order on $A^\bullet_{FY}(M)$. A Gröbner basis for $A^\bullet_{FY}(M)$ was given as follows.

Theorem 3.3.1. [FY04, Theorem 1] The following form a Gröbner basis for the ideal of $A^\bullet_{FY}(M)$:

$$\begin{cases} z_F z_G & F \text{ and } G \text{ are incomparable nonempty flats} \\ z_F \left(\sum_{H\geq G} z_H\right)^{\mathrm{rk}\, G - \mathrm{rk}\, F} & F \subsetneq G \text{ nonempty flats} \\ \left(\sum_{H\geq G} z_H\right)^{\mathrm{rk}\, G} & G \text{ a nonempty flat} \end{cases}.$$

The Gröbner basis computation in [FY04] carries over to the simplicial presentation easily. Again, pick a total ordering $>$ of $\mathscr{L}_M$ such that if $\mathrm{rk}_M(F) \leq \mathrm{rk}_M(G)$, then $F > G$.

Proposition 3.3.2. The following is a Gröbner basis for the defining ideal of $A^\bullet_\nabla(M)$ with respect to the lex monomial ordering induced by $>$:

$$\begin{cases} (\sum_{F\subseteq G} \mu(F,G) h_G)(\sum_{F'\subseteq G'} \mu(F',G') h_{G'}) & F, F' \text{ incomparable} \\ (\sum_{F\subseteq G} \mu(F,G) h_G) \cdot h_{F'}^{\mathrm{rk}\, F' - \mathrm{rk}\, F} & F \subsetneq F' \\ h_F^{\mathrm{rk}\, F} & F \in \mathcal{L}_M \setminus \{\emptyset\} \end{cases}.$$

Proof. Let $S_{FY} = \mathbb{R}[z_F : F \in \mathscr{L}_M \setminus \{\emptyset\}]$ and $S_\nabla = \mathbb{R}[h_F : F \in \mathscr{L}_M \setminus \{\emptyset\}]$, and define $\varphi : S_{FY} \to S_\nabla$ to be the substitution $z_F \mapsto -\sum_{F\subseteq G} \mu(F,G) h_G$.

Observe that φ is lower triangular with -1's on the diagonal when the variables z_F and h_F are written in descending order with respect to $>$. Hence, if $f \in S$ with initial monomial $z_{F_1}^{e_1} \cdots z_{F_k}^{e_k}$, then the initial monomial of $\varphi(f)$ is $h_{F_1}^{e_1} \cdots h_{F_k}^{e_k}$. The proposition now follows from the fact that the elements of the Gröbner basis above are the images under φ of the elements of the Gröbner basis given in Theorem 3.3.1. $\square$

As a result, we obtain a monomial basis of $A^\bullet_\nabla(M)$.

Corollary 3.3.3. For $c \in \mathbb{Z}_{\geq 0}$, a monomial $\mathbb{R}$-basis for the degree c part $A^c_\nabla(M)$ of the Chow ring $A^\bullet_\nabla(M)$ of a matroid M is

$$\{h_{F_1}^{a_1} \cdots h_{F_k}^{a_k} \mid \sum a_i = c,\ \emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k,\ 1 \leq a_i < \mathrm{rk}_M(F_i) - \mathrm{rk}_M(F_{i-1})\}.$$

We call this basis of $A^\bullet_\nabla(M)$ the **nested basis** of the Chow ring of M.

While the nested basis of $A^\bullet_\nabla(M)$ looks identical to the one given for $A^\bullet_{FY}(M)$ in [FY04, Corollary 1], we show here that with the simplicial presentation the monomials in the basis now allow for a combinatorial interpretation as a distinguished set of matroid quotients of M.

Let $f : M \twoheadrightarrow M'$ be a matroid quotient on a ground set E.

Definition 3.3.4. An f**-cyclic flat** of f is a flat $F \in \mathscr{L}_{M'}$ such that F is minimal (with respect to inclusion) among the flats $F' \in \mathscr{L}_{M'}$ such that $n_f(F') = n_f(F)$. A matroid M' is a **relative nested quotient** of M if the f-cyclic flats of M' form a chain.

Relative nested quotients are relative generalizations of (loopless) Schubert matroids:

Example 3.3.5. If $M = U_{|E|,E}$ then any matroid M' is a quotient $f : M' \twoheadleftarrow M$. In this case, the f-cyclic flats of M' are precisely the **cyclic flats** of M', which are flats F of M' such that $M'|_F$ has no coloops. Moreover, the relative nested quotients of $U_{|E|,E}$ are called **nested matroids**, which in the literature also go by (loopless) **Schubert matroids**. See [Ham17, §2.2] for more on cyclic flats and nested matroids.

The data of cyclic flats of a matroid and their ranks determine the matroid [Bry75, Proposition 2.1]. We generalize the statement to f-cyclic flats of a matroid quotient. We first need the following fact about obtaining any matroid quotient as a sequence of elementary quotients.

Lemma 3.3.6.

(1) [Hig68], [Bry86, Exercise 7.20] Any matroid quotient $f : M' \twoheadleftarrow M$ can be obtained as a sequence of elementary quotients in a canonical way called the **Higgs factorization** of f. The Higgs factorization of a quotient $f : M' \twoheadleftarrow M$ with $n_f(E) = c$ is a sequence of elementary quotients

$$M' = M_0 \overset{\mathcal{K}_1}{\twoheadleftarrow} M_1 \overset{\mathcal{K}_2}{\twoheadleftarrow} \cdots \overset{\mathcal{K}_c}{\twoheadleftarrow} M_c = M$$

where the bases of M_i for $i = 1, \ldots, c$ are defined as

$$\mathcal{B}(M_i) = \{A \subseteq E \mid A \text{ spanning in } M', \text{ independent in } M, \text{ and } |A| = \mathrm{rk}(M') + i\}.$$

(2) [KK78, Theorem 3.4] The modular cuts $\mathcal{K}_i$ of the Higgs factorization are

$$\mathcal{K}_i = \{G \in \mathscr{L}_{M_i} \mid n_f(G) \geq i\}.$$

Proposition 3.3.7. The data of the f-cyclic flats, their f-nullities, and the matroid M determine the quotient $f : M' \twoheadleftarrow M$. More precisely, writing $n_f(E) = c$, the data recovers the Higgs factorization $M' = M_0 \overset{\mathcal{K}_1}{\twoheadleftarrow} M_1 \overset{\mathcal{K}_2}{\twoheadleftarrow} \cdots \overset{\mathcal{K}_c}{\twoheadleftarrow} M_c = M$ of f by specifying the modular cuts $\mathcal{K}_i$ to be

$$\mathcal{K}_i = \{G \in \mathscr{L}_{M_i} \mid G \supseteq F \text{ for some } F \in \mathrm{cyc}(f) \text{ with } n_f(F) \geq i\}$$

for each $i = 1, \ldots, c$.

Proof. Lemma 3.3.6.(2) implies that the modular cut $\mathcal{K}_i$ is $\{G \in \mathscr{L}_{M_i} \mid n_f(G) \geq i\}$ for each $i = 1, \ldots, c$. This can equivalently be written as $\{G \in \mathscr{L}_{M_i} \mid G \supseteq F \text{ for some } F \in \operatorname{cyc}(f) \text{ with } n_f(F) \geq i\}$ by the definition of f-cyclic flats. □

We now show that the nested basis of $A^{\bullet}_{\nabla}(M)$ given in Corollary 3.3.3 is in bijection with the set of relative nested quotients of M.

Theorem 3.3.8. Let M be a loopless matroid of rank $r = d + 1$. For each $0 \leq c \leq d$, the cap product map

$$A^c_{\nabla}(M) \to \mathrm{MW}_{d-c}(\Sigma_M), \quad \xi \mapsto \xi \cap \Delta_M$$

induces a bijection between the monomial basis for $A^c_{\nabla}(M)$ given in Corollary 3.3.3 and the set of Bergman classes $\Delta_{M'}$ of loopless relative nested quotients $M' \twoheadleftarrow M$ with $\operatorname{rk}(M') = \operatorname{rk}(M) - c$.

Proof. Let $h_{F_1}^{a_1} \cdots h_{F_k}^{a_k}$ be an element of the monomial basis given in Corollary 3.3.3. By repeated application of Theorem 3.2.3, we have

$$h_{F_1}^{a_1} \cdots h_{F_k}^{a_k} \cap \Delta_M = \Delta_{M'}$$

where

$$M' = M \textstyle\bigwedge^{a_k} H_{F_k} \cdots \bigwedge^{a_1} H_{F_1} := M \wedge \underbrace{H_{F_k} \wedge \cdots \wedge H_{F_k}}_{a_k \text{ times}} \wedge \cdots \wedge \underbrace{H_{F_1} \wedge \cdots \wedge H_{F_1}}_{a_1 \text{ times}}.$$

By Lemma 3.2.4, we can consider the matroid M' to be the result of a sequence of principal truncations on M, first by F_k repeated a_k times, then by F_{k-1} repeated a_{k-1} times, and so forth. Taking a principal truncation by F_i in this process makes sense due to the following observation: The description of the flats in a principal truncation (Proposition 3.1.5) implies that the inequalities on the a_i's in Corollary 3.3.3 ensure that F_i is a flat in $M \bigwedge^{a_k} H_{F_k} \cdots \bigwedge^{a_{i+1}} H_{F_{i+1}}$, and in particular $\operatorname{rk}(F_1) - a_1 > 0$ ensures loopless.

The flats that decrease in rank under a principal truncation by F_i are exactly those that contain F_i. Hence, our consideration of M' as the sequence of principal truncations implies that $f : M' \twoheadleftarrow M$ is a matroid quotient with $\operatorname{cyc}(f) = \{F_1, \ldots, F_k\}$ and $n_f(F_j) = \sum_{i=1}^{j} a_i$. We have thus shown that an element of the nested basis defines a relative nested quotient by the cap product.

Conversely, Proposition 3.3.7 implies that the f-cyclic flats and their f-nullities of a relative nested quotient $f : M' \twoheadleftarrow M$ recovers the Higgs factorization

$$M' = M_0 \overset{\mathcal{K}_1}{\twoheadleftarrow} M_1 \overset{\mathcal{K}_2}{\twoheadleftarrow} \cdots \overset{\mathcal{K}_c}{\twoheadleftarrow} M_c = M$$

of f by specifying the modular cuts to be

$$\mathcal{K}_i = \{G \in \mathscr{L}_{M_i} \mid G \supseteq F \text{ for some } F \in \operatorname{cyc}(f) \text{ with } n_f(F) \geq i\}.$$

Thus, if $\operatorname{cyc}(f) = \{F_1 \subsetneq \cdots \subsetneq F_k\}$, then $M' = M \bigwedge^{a_k} H_{F_k} \cdots \bigwedge^{a_1} H_{F_1}$ where $a_j = n_f(F_j) - n_f(F_{j-1})$ for $j > 1$, and $a_1 = n_f(F_1)$. □

Moreover, the bijection given in the previous theorem respects linear independence.

Proposition 3.3.9. The elements

$$\{\Delta_{M'} \ : \ M' \text{ is a loopless relative nested quotient of } M\}$$

are linearly independent in $\mathrm{MW}_\bullet(\Sigma_{A_n})$.

The proof given below is a modification of the one given for nested matroids in [Ham17, Proposition 3.2].

Proof. For a loopless nested matroid quotient $f : M' \twoheadleftarrow M$ of corank $c = \mathrm{rk}(M) - \mathrm{rk}(M')$ given by a sequence of $(F_0 := \emptyset, r_0 := 0), (F_1, r_1), \ldots, (F_k, r_k)$ of the f-cyclic flats and their ranks, define

$$\gamma(f) := (d_i)_{i=1,\ldots,r}, \qquad d_i := \begin{cases} r_i - r_{i-1} & \text{if } i \leq k \\ 0 & \text{otherwise} \end{cases}.$$

Denote by $\mathscr{M}_r$ the set of loopless nested matroid quotients $f : M' \twoheadleftarrow M$ of rank $\mathrm{rk}(M') = r$. Assume that we have a linear relation

$$\sum_{f:M' \twoheadleftarrow M \in \mathscr{M}_r} a_{M'} \Delta_{M'} = 0.$$

We show by lexicographic induction on $\gamma(f)$ that $a_{M'} = 0 \ \forall M' \in \mathscr{M}_r$.

For the base case, consider the case of $f : M' \twoheadleftarrow M$ with f-cyclic flats $\emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k$ satisfying

$$\gamma(f) = (\underbrace{1, \ldots, 1}_{k \text{ many}}, 0, \ldots, 0).$$

Extend the chain of cyclic flats of M' to any maximal chain of flats in M', and consider a loopless nested matroid quotient $g : N' \twoheadleftarrow M$ also containing this chain as a maximal chain of flats. We show that $N' = M'$. Note that by construction $\mathrm{rk}_{N'}(F_i) = \mathrm{rk}_{M'}(F_i)$ for all $0 \leq i \leq k$. By induction assume $F_0, \ldots, F_{j-1}$ are g-cyclic. If F_j is not g-cyclic, then it contains a g-cyclic flat G with the same g-nullity as that of F_j. But then $n_g(G) = n_g(F_j) = n_f(F_j) > n_f(F_{j-1}) = n_g(F_{j-1})$, implies $G \supsetneq F_{j-1}$, which contradicts $\mathrm{rk}_{N'}(G) < \mathrm{rk}_{N'}(F_j) = \mathrm{rk}_{N'}(F_{j-1}) + 1$. Thus, all F_i's are g-cyclic as well with $\mathrm{rk}_{N'}(F_i) = i$, and there are are no other g-cyclic flats since $n_f(E) = n_g(E)$.

Now suppose $\gamma(f) = (d_1, \ldots, d_r) > (0, \ldots, 0)$ and consider $g : N' \twoheadleftarrow M$ that has a maximal chain of flats that is also a maximal chain in $\mathscr{L}_{M'}$ containing the f-cyclic flats. We show that if $N' \neq M'$ then $\gamma(g) <_{lex} \gamma(f)$, thereby completing the induction to conclude that $a_{M'} = 0 \ \forall M'$.

Let $\gamma(g) = (c_1, \ldots, c_r)$, and suppose $1 \leq j \leq k$ is the minimum j such that F_j is not g-cyclic, which exists since $N' \neq M'$. By the same arguments given in the case of $\gamma(f) = (1, \ldots, 1, 0, \ldots, 0)$, we then have a g-cyclic flat G such that $F_{j-1} \subsetneq G \subsetneq F_j$, which decreases c_j by at least one. Hence, $\gamma(g) < \gamma(f)$, as desired. □

Chapter 4

The Poincaré duality property

In this chapter, we recover the first component of the Kähler package for the Chow ring of a matroid, called the Poincaré duality property. We review some facts about Poincaré duality algebras in §4.1, but we will only need a small portion of these facts in this chapter—the rest will be needed only later in Chapter 6. We establish the Poincaré duality property of Chow rings of matroids and discuss some consequences in §4.2

4.1 Poincaŕe duality algebras and their transports

We review some general algebraic notions about Poincaré duality algebras. Let $\mathbb{k}$ be a field.

Definition 4.1.1. A graded finite (commutative) $\mathbb{k}$-algebra $A^\bullet = \bigoplus_{i=0}^{d} A^i$ is a **(graded) Poincaré duality algebra of dimension** d if (i) $A^0 = \mathbb{k}$, and (ii) there exists an isomorphism $\int : A^d \xrightarrow{\sim} \mathbb{k}$, called the **degree map** of $A^\bullet$, such that the map

$$A^k \to \operatorname{Hom}(A^{d-k}, \mathbb{k}), \quad \xi \mapsto (\zeta \mapsto \int \xi \cdot \zeta)$$

is an isomorphism for all $0 \le k \le d$, or equivalently, the pairing

$$A^i \times A^{d-i} \to A^d \simeq \mathbb{k}, \quad (\xi, \zeta) \mapsto \int \xi \cdot \zeta$$

is a non-degenerate for all $0 \le i \le d$.

We write $(A^\bullet, \int)$ for a Poincaré duality algebra with a chosen degree map $\int$. In Chapter 6, we will often drop the degree symbol $\int$ when the context is clear; in particular, for $\zeta \in A^1$ we will often write ζ^d to mean $\int \zeta^d$.

We will use the following construction to establish that Chow rings of matroids are Poincaré duality algebras.

Proposition 4.1.2. If $(A^\bullet, \int)$ is a Poincaré duality algebra of dimension d, and $f \in A^\bullet$ a homogeneous element of degree k. Then the $\Bbbk$-algebra

$$A^\bullet / \operatorname{ann}(f), \quad \text{where } \operatorname{ann}(f) = \{a \in A^\bullet \mid af = 0\}$$

is a Poincaré duality algebra of dimension $d-k$ with the **induced degree map** $\int_f$ defined by $\int_f (a + \operatorname{ann}(f)) := \int af$ for $a \in A^{d-k}$.

Proof. This is a straightforward check; see [MS05, Corollary I.2.3.] for example. □

It will sometimes be convenient to identify elements of the ring $A^\bullet / \operatorname{ann}(f)$ to elements of the principal ideal $\langle f \rangle \subset A^\bullet$, with multiplication is by $af \cdot bf = (ab) \cdot f$. The construction in Proposition 4.1.2 will arise in next subsection §4.2 with f being the Bergman class.

The rest of this subsection on Poincaré duality algebras will not be needed until Chapter 6.

We describe another way the construction in Proposition 4.1.2 arises in the context of Chow cohomology rings of fans. Let Σ be a d-dimensional smooth rational fan in $N_\mathbb{R}$ for a lattice N, and let $\rho \in \Sigma(1)$ be ray. Denote by $\overline{u}$ the image of $u \in N_\mathbb{R}$ under the projection $N_\mathbb{R} \twoheadrightarrow N_\mathbb{R} / \operatorname{span}(\rho)$. The **star of Σ at** ρ is a $(d-1)$-dimensional fan in $N_\mathbb{R} / \operatorname{span}(\rho)$ defined by

$$\operatorname{star}(\rho, \Sigma) := \{\overline{\sigma} \mid \sigma \in \Sigma \text{ contains } \rho\}.$$

By definition of the Chow cohomology ring, one can check that there is a surjection $A^\bullet(\Sigma) \twoheadrightarrow A^\bullet(\operatorname{star}(\rho, \Sigma))$ determined by

$$x_{\rho'} \mapsto \begin{cases} x_{\overline{\rho'}} & \text{if } \rho' \text{ and } \rho \text{ form a cone in } \Sigma \\ 0 & \text{otherwise} \end{cases} \qquad \text{for each } \rho' \neq \rho.$$

Since $\langle x_{\rho'} \mid \rho' \text{ and } \rho \text{ do not form a cone in } \Sigma \rangle \subset \operatorname{ann}_{A^\bullet(\Sigma)}(x_\rho)$, thus we get an induced map

$$\pi_\rho : A^\bullet(\operatorname{star}(\rho, \Sigma)) \twoheadrightarrow A^\bullet(\Sigma) / \operatorname{ann}(x_\rho).$$

Geometrically, a ray ρ corresponds to a torus-invariant divisor $V(\rho)$ of the toric variety X_Σ via the orbit-cone correspondence [CLS11, Theorem 3.2.6], and the toric variety $X_{\operatorname{star}(\rho,\Sigma)}$ of the star is the subvariety $V(\rho)$. The map $A^\bullet(\Sigma) \to A^\bullet(\operatorname{star}(\rho, \Sigma))$ described above is the pullback map of algebraic cycles along the closed embedding $X_{\operatorname{star}(\rho,\Sigma)} \simeq V(\rho) \hookrightarrow X_\Sigma$.

In Chapter 6, we will use the following criterion for when the map π_ρ is an isomorphism.

Proposition 4.1.3. [AHK18, Proposition 7.13] Suppose that the Chow cohomology ring $A^\bullet(\Sigma)$ is a Poincaré duality algebra. Then, the map $\pi_\rho : A^\bullet(\mathrm{star}(\rho,\Sigma)) \twoheadrightarrow A^\bullet(\Sigma)/\operatorname{ann}(x_\rho)$ is an isomorphism if and only if $A^\bullet(\mathrm{star}(\rho,\Sigma))$ is a Poincaré duality algebra.

Proof. The algebra $A^\bullet(\Sigma)/\operatorname{ann}(x_\rho)$ is a Poincare duality algebra by Proposition 4.1.2. The statement thus follows from Proposition 4.1.4.(2) below, which states that a surjection of Poincare duality algebras of same dimension is an isomorphism. $\square$

Two useful facts about Poincaré duality algebras follow. Both are straightforward to check.

Proposition 4.1.4. Let $(A^\bullet, \int_A)$ and $(B^\bullet, \int_B)$ be Poincaré duality algebras of dimension d_A and d_B over a common field $\Bbbk$.

(1) The tensor product $(A \otimes B)^\bullet = \bigoplus_\bullet \left(\bigoplus_{i+j=\bullet} A^i \otimes B^j\right)$ is also a Poincaré duality algebra of dimension $d_A + d_B$ with degree map

$$\int_{A\otimes B} : (A \otimes B)^{d_A+d_B} = A^{d_A} \otimes B^{d_B} \to \Bbbk, \quad a \otimes b \mapsto \int_A a \cdot \int_B b.$$

(2) A surjection $A^\bullet \twoheadrightarrow B^\bullet$ of Poincaré duality algebras of the same dimension (that is, if $d_A = d_B$) is an isomorphism.

4.2 Poincaré duality for matroids

We show that the Chow ring $A^\bullet(M)$ of a loopless matroid M is a Poincaré duality algebra with $\int_M$ as the degree map. While this was proved in [AHK18, Theorem 6.19], we give a non-inductive proof by using the simplicial presentation that avoids *flips*, a technical tool in [AHK18] which introduces combinatorial objects from outside the realm of matroids.

Our main theorem of the section is the following.

Theorem 4.2.1. Let M be a loopless matroid of rank $r = d + 1$ on a ground set $E = \{0, 1, \ldots, n\}$, and consider the Bergman class $\Delta_M \in \mathrm{MW}_d(\Sigma_{A_n})$ as an element of $A^\bullet(\Sigma_{A_n})$ via the isomorphism $A^\bullet(\Sigma_{A_n}) \simeq \mathrm{MW}_{n-\bullet}(\Sigma_{A_n})$ in Theorem 2.1.6. Then, we have

$$A^\bullet(M) \simeq A^\bullet(\Sigma_{A_n})/\operatorname{ann}(\Delta_M).$$

Since $A^\bullet(\Sigma_{A_n})$ is a Poincare duality algebra (Theorem 2.1.6), Proposition 4.1.2 immediately implies the following corollary.

Corollary 4.2.2. The Chow ring $A^\bullet(M)$ is a graded Poincaré duality algebra of dimension $\mathrm{rk}(M) - 1$ with $\int_M$ as the degree map.

Remark 4.2.3. Because Σ_{A_n} is a smooth projective fan, there exist a purely combinatorial proof of the Poincaré duality for its Chow cohomology ring via the line shelling of the fan; see [FK10]. While Bergman fans of matroids are also shellable [Bjo92], they are not complete (and hence not projective), and the arguments of [FK10] does not readily modify to give Poincaré duality for Chow rings of matroids.

We also obtain the following generalization of [Ham17, Corollary 3.13].

Corollary 4.2.4. For each $0 \le c \le d$, the cap product map

$$A^c(M) \to \mathrm{MW}_{d-c}(\Sigma_M), \quad \xi \mapsto \xi \cap \Delta_M$$

is an isomorphism of $\mathbb{R}$-vector spaces. Thus, the Bergman classes of relative nested quotients form a basis of $\mathrm{MW}_\bullet(\Sigma_M)$.

Proof. The first statement follows from Corollary 4.2.2 and the fact that $\mathrm{Hom}(A^c(M), \mathbb{R}) \simeq \mathrm{MW}_{d-c}(\Sigma_M)$ (Lemma 2.1.5). Theorem 3.3.8 then implies the second statement. $\square$

Proof of Theorem 4.2.1. Recall that the isomorphism $A^\bullet(\Sigma_{A_n}) \simeq \mathrm{MW}_{n-\bullet}(\Sigma_{A_n})$ makes the set of Minkowski weights into a graded ring, denoted $\mathrm{MW}^\bullet(\Sigma_{A_n})$. Let ι_M^* be the pullback map of the inclusion $\iota_M : \Sigma_M \hookrightarrow \Sigma_{A_n}$. A formal property of cap products, given below in Lemma 4.2.5, gives us a commuting diagram of surjections

$$\begin{tikzcd}
A^\bullet(\Sigma_{A_n}) \arrow[r, "\sim"] \arrow[d, two heads, "\iota_M^*"'] & \mathrm{MW}^\bullet(\Sigma_{A_n}) \arrow[d, two heads] \\
A^\bullet(M) \arrow[r, two heads] & \mathrm{MW}^\bullet(\Sigma_{A_n}) / \operatorname{ann}(\Delta_M)
\end{tikzcd}
\qquad
\begin{tikzcd}
\zeta \arrow[r, mapsto] \arrow[d, mapsto] & \zeta \cap \Delta_{\Sigma_{A_n}} \arrow[d, mapsto] \\
\iota_M^*\zeta \arrow[r, mapsto] & \zeta \cap \Delta_M.
\end{tikzcd}$$

(Here we have identified the elements of $\mathrm{MW}^\bullet(\Sigma_{A_n}) / \operatorname{ann}(\Delta_M)$ with the elements of the principal ideal $\langle \Delta_M \rangle \subset \mathrm{MW}^\bullet(\Sigma_{A_n})$.) Proposition 3.3.9 states that the bottom horizontal map preserves linear independence, and hence is injective. $\square$

Lemma 4.2.5. Let ι^* be the pullback map of an inclusion of fans $\iota : \Sigma' \hookrightarrow \Sigma$ where Σ is complete, and let $\delta_\Sigma : A^\bullet(\Sigma) \xrightarrow{\sim} \mathrm{MW}^\bullet(\Sigma)$ be the isomorphism in Theorem 2.1.6. Suppose Δ is a Minkowski weight on Σ whose support $|\Delta|$ is contained the support $|\Sigma'|$. Then we have a diagram

$$\begin{tikzcd}
A^\bullet(\Sigma) \arrow[r, "\sim"] \arrow[d, two heads, "\iota^*"'] & \mathrm{MW}^\bullet(\Sigma) \arrow[d, two heads] \\
A^\bullet(\Sigma') \arrow[r, two heads] & \mathrm{MW}^\bullet(\Sigma) / \operatorname{ann}(\Delta)
\end{tikzcd}
\qquad
\begin{tikzcd}
\zeta \arrow[r, mapsto] \arrow[d, mapsto] & \zeta \cap \Delta_\Sigma \arrow[d, mapsto] \\
\iota^*\zeta \arrow[r, mapsto] & \zeta \cap \Delta.
\end{tikzcd}$$

Proof. We need show that the kernel $\langle x_\rho \mid \rho \in \Sigma(1) \setminus \Sigma'(1)\rangle \subset A^\bullet(\Sigma)$ of the pullback map ι^* is contained in the kernel of the map $A^\bullet(\Sigma) \xrightarrow{\sim} \mathrm{MW}^\bullet(\Sigma) \twoheadrightarrow \mathrm{MW}^\bullet(\Sigma)/\operatorname{ann}(\Delta)$. Since $|\Delta| \subseteq |\Sigma'|$, we may consider Δ as a Minkowski weight on Σ', and thus by functoriality of the cap product (Remark 2.1.8), we have $x_\rho \cap \Delta = \iota^* x_\rho \cap \Delta = 0$ for $x_\rho \in A^\bullet(\Sigma)$ where $\rho \in \Sigma(1) \setminus \Sigma'(1)$. □

Remark 4.2.6 (cf. Phenomenon I)**.** Let $Y_{\mathscr{R}(M)}$ be the wonderful compactification of a realization of M. Since the pullback map $A^\bullet(X_{\Sigma_{A_n}}) \to A^\bullet(Y_{\mathscr{R}(M)})$ along the closed embedding $Y_{\mathscr{R}(M)} \hookrightarrow X_{A_n}$ is surjective (Remark 2.3.3), we have $A^\bullet(M) \simeq A^\bullet(Y_{\mathscr{R}(M)}) \simeq A^\bullet(X_{\Sigma_{A_n}})/\operatorname{ann}([Y_{\mathscr{R}(M)}])$.

Remark 4.2.7. Let M be a loopless of rank $d = r+1 < n+1$. Then the isomorphism $A^\bullet(M) \simeq \mathrm{MW}_{d-\bullet}(\Sigma_M)$ that we have established makes $\mathrm{MW}^\bullet(\Sigma_M) := \mathrm{MW}_{d-\bullet}(\Sigma_M)$ into a graded ring. We caution that the resulting multiplication structure is *not* the usual stable intersection of Minkowski weights.

Chapter 5

Log-concavity of the volume polynomial

A presentation of a graded Poincaré duality algebra $A^\bullet$ can be encoded via the Macaulay inverse system of commutative algebra into a single polynomial VP_A called the volume polynomial of $A^\bullet$ [Eis95, §21.2]. In geometric contexts, the volume polynomial takes on an additional meaning: if $A^\bullet$ is the ring of algebraic cycles modulo numerical equivalence of a smooth projective variety, then VP_A measures degrees of ample divisors (see [Ein+05] for a survey), and if $A^\bullet$ is the Chow cohomology ring of a complete smooth fan Σ, then VP_A measures the volumes of polytopes whose normal fans coarsen Σ [CLS11, §13]. Furthermore, in both geometric contexts, the volume polynomial of $A^\bullet$ is positive and log-concave on the ample cone when considered as a function $A^1 \to \mathbb{R}$ [Laz04, Corollary 1.6.3.(iii)].

In this chapter, we give a combinatorial formula for the volume polynomial VP_M^∇ of the Chow ring $A_\nabla^\bullet(M)$ of a loopless matroid M, and show that, as in the geometric cases, the volume polynomial VP_M^∇ when regarded as a function $A_\nabla^1(M) \to \mathbb{R}$ is both positive and log-concave on a subcone $\mathscr{K}_M^\nabla$ of the ample cone $\mathscr{K}_M$ generated by the simplicial generators. While the results of [AHK18] imply that the volume polynomial of a matroid satisfies such properties, we give here an independent and more direct proof by establishing that VP_M^∇ is a Lorentzian polynomial as defined in [BH19]. In the next Chapter 6, we build upon the results of this chapter to conclude that VP_M^∇ is both positive and log-concave on the entire ample cone $\mathscr{K}_M$.

5.1 Volume polynomials and Lorentzian polynomials

Here we review the notion of volume polynomials and how they generalize to Lorentzian polynomials.

One can encode a graded Poincaré duality algebra into a single polynomial called the volume polynomial as follows.

Definition 5.1.1. Let $(A^\bullet, \int)$ be a graded Poincaré duality algebra of dimension d that is generated in degree 1, with a chosen presentation $A^\bullet = \mathbb{k}[x_1, \dots, x_s]/I$ and a degree map $\int : A^d \to \mathbb{k}$. Then its **volume polynomial** VP_A is a multivariate polynomial in $\mathbb{k}[t_1, \dots, t_s]$ defined by

$$VP_A(t_1, \dots, t_s) := \int (t_1 x_1 + \cdots + t_s x_s)^d$$

where we extend the degree map $\int$ to $A[t_1, \dots, t_s] \to \mathbb{k}[t_1, \dots, t_s]$.

If $(A^\bullet, \int)$ is a Poincaré duality algebra with a presentation $A^\bullet = \mathbb{k}[x_1, \dots, x_s]/I$, then the defining ideal I can be recovered from the volume polynomial VP_A as follows [CLS11, Lemma 13.4.7]

$$I = \{ f(x_1, \dots, x_s) \in \mathbb{k}[x_1, \dots, x_s] \mid f(\tfrac{\partial}{\partial t_1}, \dots, \tfrac{\partial}{\partial t_s}) \cdot VP_A(t_1, \dots, t_s) = 0 \}.$$

In [BH19], the authors define Lorentzian polynomials as a generalization of volume polynomials in algebraic geometry and stable polynomials in optimization. Here we briefly summarize the relevant results.

Definition 5.1.2. A homogeneous polynomial $f \in \mathbb{R}[x_1, \dots, x_n]$ of degree d is **strictly Lorentzian** if its support consists of all monomials in x of degree d, all of its coefficients are positive, and any of its $(d-2)$-th order partial differentiation $\partial_{i_1} \cdots \partial_{i_{d-2}} f$ has Hessian matrix with Lorentzian signature $(+, -, -, \dots, -)$. **Lorentzian polynomials** are polynomials that can be obtained as a limit of strictly Lorentzian polynomials.

To characterize Lorentzian polynomials, we need a combinatorial notion that mirrors the exchange axiom for matroids: a collection of points $J \subset \mathbb{Z}^n_{\geq 0}$ is **M-convex** if for any $\alpha, \beta \in J$ and $i \in [n]$ with $\alpha_i > \beta_i$ there exists $j \in [n]$ such that $\alpha_j < \beta_j$ and $\alpha - e_i + e_j \in J$. When the elements of J all have the same coordinate sum, this is equivalent to stating that the convex hull of J is a generalized permutohedra [Mur03, Theorem 1.9].

The following characterization can be seen as a linear algebraic abstraction of the proof of Teissier-Khovanskii inequalities via the Hodge index theorem for algebraic surfaces.

Theorem 5.1.3. [BH19, Theorem 5.1] A homogeneous polynomial $f \in \mathbb{R}[x_1, \dots, x_n]$ of degree d with nonnegative coefficients is Lorentzian if and only if the following two conditions are satisfied:

(1) The support of f is M-convex, and

(2) The Hessian matrix of $\partial_{i_1} \cdots \partial_{i_{d-2}} f$ has at most one positive eigenvalue for any choice of $(d-2)$th order partial differentiation.

Proposition 5.1.4. We note some operations that preserve the Lorentzian property of polynomials.

(1) [BH19, Corollary 5.5] Products of Lorentzian polynomials are Lorentzian.

(2) [BH19, Theorem 2.10] If a polynomial $f \in \mathbb{R}[x_1, \dots, x_n]$ is Lorentzian, then so is $f(Ax) \in \mathbb{R}[x_1, \dots, x_m]$ for any $n \times m$ matrix A with non-negative entries.

Implications to log-concavity phenomena in combinatorics arise from the following properties of Lorentzian polynomials.

Theorem 5.1.5. Let $f \in \mathbb{R}[x_1, \dots, x_n]$ be a homogeneous polynomial with nonnegative coefficients. The Lorentzian property of f can be characterized via log-concavity properties as follows.

(1) [BH19, Theorem 5.3] A homogeneous polynomial f is Lorentzian if and only if f is strongly log-concave, in the sense that if g is any partial derivative of f of any order, then either g is identically zero or $\log g$ is concave on the positive orthant $\mathbb{R}^n_{>0}$.

(2) [BH19, Example 5.2] If $n = 2$, so that $f = \sum_{k=0}^d a_k x_1^k x_2^{d-k}$, then f is Lorentzian if and only if $(a_0, a_1, \dots, a_d)$ has no internal zeroes and is ultra log-concave, that is,

$$a_{k_1} a_{k_3} \neq 0 \implies a_{k_2} \neq 0 \text{ for all } 0 \leq k_1 < k_2 < k_3 \leq d, \text{ and}$$

$$\frac{a_k^2}{\binom{d}{k}^2} \geq \frac{a_{k-1} a_{k+1}}{\binom{d}{k-1}\binom{d}{k+1}} \text{ for all } 0 < k < d.$$

We remark that (strictly) Lorentzian polynomials arise in classical algebraic geometry whenever one has a set of nef (ample) divisors on a smooth projective variety.

Remark 5.1.6. Let $\{D_1, \dots, D_s\}$ be nef (ample) divisors on a smooth projective $\mathbb{k}$-variety X of dimension d, and $A(X)$ its Chow ring. Let $\int_X : A^d(X) \to \mathbb{R}$ be the degree map obtained as the pushforward map along the structure map $X \to \operatorname{Spec} \mathbb{k}$. Then

$$\operatorname{vol}_X \Big(\sum_{i=1}^s t_i D_i \Big) := \lim_{q \to \infty} \frac{\dim_{\mathbb{k}} H^0(q \sum_i t_i D_i)}{q^d / d!} = \int_X \Big(\sum_i t_i D_i \Big)^d$$

is a (strictly) Lorentzian polynomial [Laz04, Corollary 1.6.3.(iii)] or [BH19, Theorem 10.1].

5.2 The Dragon Hall-Rado formula

We prepare our formula for the volume polynomial of $A^\bullet_\nabla(M)$ by describing the combinatorial notions in [Pos09] that we generalize to arbitrary matroids.

We first recall Hall's marriage theorem and Rado's generalization; for proofs we point to [Oxl11, §11.2]. Let $E = [n] = \{0, 1, \dots, n\}$. A **transversal** of a collection $\{A_0, \dots, A_m\}$

(repetitions allowed) of subsets of E is a subset $I \subseteq E$ such that there exists a bijection $\phi : \{A_0, \dots, A_m\} \to I$ satisfying $\phi(A_i) \in A_i$ for all $0 \le i \le m$. Hall considered the following problem:

Given $A_0, \dots, A_n \subseteq E$, when does $\{A_0, \dots, A_n\}$ have a transversal?

The well-known Hall's marriage theorem [Hal35] answers this problem: a transversal exists for $\{A_0, \dots, A_n\}$ if and only if $\left|\bigcup_{j \in J} A_i\right| \ge |J|$ for all $J \subseteq [n]$. The following theorem of Rado gives a matroid generalization of the condition given in Hall's theorem.

Theorem 5.2.1 (Rado's theorem [Rad42]). Let M be a matroid on E. A family of subsets $\{A_0, \dots, A_m\}$ of E has a transversal $I \subseteq E$ that is independent in M if and only if

$$\mathrm{rk}_M \left(\bigcup_{j \in J} A_j \right) \ge |J|, \quad \forall J \subseteq [m].$$

Hall's condition can be recovered from Rado's by setting $M = U_{|E|,E}$ and $m = n$. See [Oxl11, Theorem 11.2.2] for more information and a proof of Rado's theorem. The following variant of Hall's marriage theorem was investigated by Postnikov as a combinatorial interpretation of a formula for volumes of generalized permutohedra [Pos09, §5, §9].

Proposition 5.2.2 (Dragon marriage condition). Let $\{A_1, \dots, A_n\}$ be a collection of subsets of $E = \{0, 1, \dots, n\}$. There is a transversal $I \subseteq E \setminus \{e\}$ of $\{A_1, \dots, A_n\}$ for every $e \in E$ if and only if

$$\left| \bigcup_{j \in J} A_j \right| \ge |J| + 1, \quad \forall \varnothing \subsetneq J \subseteq \{1, 2, \dots, n\}.$$

The dragon marriage theorem above follows easily from the original Hall's marriage theorem, and conversely, one can obtain Hall's marriage theorem from the dragon marriage theorem as follows: given $A_0, \dots, A_n \subseteq E$ as in Hall's theorem, set $E' = E \sqcup \{*\}$ and $A'_i := A_0 \sqcup \{*\}$ for each $0 \le i \le n$ and apply Postnikov's theorem to $\{A'_0, \dots, A'_n\}$.

We now consider a variant of Rado's theorem in the same spirit.

Proposition 5.2.3 (Dragon Hall-Rado condition). Let M be a matroid on E, and let $\{A_1, \dots, A_m\}$ be a collection of subsets of E. Then there is a transversal $I \subseteq E \setminus \{e\}$ of $\{A_1, \dots, A_m\}$ for every $e \in E$ if and only if

$$\mathrm{rk}_M \left(\bigcup_{j \in J} A_j \right) \ge |J| + 1, \quad \forall \varnothing \subsetneq J \subseteq \{1, \dots, m\}$$

and when this condition is satisfied, we say that $\{A_1, \dots, A_m\}$ satisfy the **dragon Hall-Rado condition of** M, or $\mathrm{DHR}(M)$ for short.

Proof. This follows from Theorem 5.2.1 and the observation that independent transversals $I \subseteq E \setminus \{e\}$ of $\{A_1, \ldots, A_m\}$ are the same as independent transversals of $\{A_1 \setminus \{e\}, \ldots, A_m \setminus \{e\}\}$. $\square$

We can obtain Rado's theorem from the dragon Hall-Rado theorem by an argument analogous to how Hall's marriage theorem is obtained from dragon marriage theorem. In summary, the combinatorics introduced in this subsection thus far can be schematically laid out as follows with the indicated logical implications:

$$\begin{array}{ccc} \text{Hall's marriage theorem} & \Longleftarrow\!\!=\!\!= & \text{Rado's theorem} \\ \Updownarrow & & \Updownarrow \\ \text{Dragon marriage theorem} & \Longleftarrow & \text{Dragon Hall-Rado theorem} \end{array}$$

We are now ready to compute the intersection numbers of the variables h_F in the $A^\bullet_\nabla(M)$ presentation of the Chow ring of a matroid M.

Theorem 5.2.4. Let $A_1, \ldots, A_d$ be a collection of subsets of E, and M a loopless matroid on E of rank $d+1$. Let $H_{A_1}, \ldots, H_{A_d}$ be matroids as defined in Proposition 3.2.4. Then

$$M \wedge H_{A_1} \wedge \cdots \wedge H_{A_d} = U_{1,E} \iff \{A_1, \ldots, A_d\} \text{ satisfy DHR}(M).$$

Thus, we have

$$\int_M h_{A_1}(M) \cdots h_{A_d}(M) = \begin{cases} 1 & \text{if } \{A_1, \ldots, A_d\} \text{ satisfy DHR}(M) \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For the first assertion, we begin by making two easy observations.

1. $M \wedge H_S$ has a loop if and only if $\mathrm{rk}_M(S) = 1$, and
2. [Oxl11, Exercise 7.3.10] for the elementary quotient $f : M \twoheadrightarrow M \wedge H_S$ we have by Proposition 3.1.5 that

$$\{T \subseteq E \mid n_f(T) = 1\} = \{T \subseteq E \mid \mathrm{cl}_M(T) \supseteq S\}.$$

For the necessity of the condition, note that if $\mathrm{rk}_M(\bigcup_{j \in J} A_j) \leq k$ for some $J = \{j_1, \ldots, j_k\} \subseteq \{1, \ldots, d\}$ ($k > 0$), then for $\widetilde{M} := M \wedge H_{A_{j_1}} \wedge \cdots \wedge H_{A_{j_{k-1}}}$ we have $\mathrm{rk}_{\widetilde{M}}(\bigcup_{j \in J} A_j) \leq k - (k-1) = 1$, so that $M \bigwedge_{j \in J} H_{A_j}$ already has a loop.

For sufficiency, we induct on d. The base case $d = 1$ is trivially satisfied. Now, we claim that if $\{A_1, \ldots, A_d\}$ satisfy the dragon Hall-Rado condition for M, then so does $\{A_1, \ldots, A_{d-1}\}$ for $\widetilde{M} := M \wedge N_{A_d}$. For the sake of contradiction, suppose $\mathrm{rk}_{\widetilde{M}}(A_1 \cup \cdots \cup A_k) \leq k$, then we must have had $\mathrm{rk}_M(A_1 \cup \cdots \cup A_k) = k+1$ with $\mathrm{cl}_M(A_1 \cup \cdots \cup A_k) \supseteq A_d$. But then $\mathrm{rk}_M(A_1 \cup \cdots \cup A_k \cup A_d) = k+1$, violating DHR$(M)$.

For the second assertion, we note that $\int_M h_{A_1}(M)\cdots h_{A_d}(M) = \int_{\Sigma_{A_n}} h_{A_1}\cdots h_{A_d} \cap \Delta_M$, and the (unique) loopless matroid $U_{1,E}$ of rank 1 on E defines the Bergman class $\Delta_{U_{1,E}}$ by $\Delta_{U_{1,E}}(\mathbf{0}) = 1$, where $\mathbf{0}$ is the zero-dimensional cone of Σ_{A_n}, so that $\int_{\Sigma_{A_n}}(\Delta_{U_{1,E}}) = 1$. $\square$

We obtain as an immediate corollary the promised generalization of [Pos09, Corollary 9.4]. Recall that $\mathscr{L}_M^{\geq 2}$ denotes the flats of M of rank at least two.

Corollary 5.2.5. Let M be a loopless matroid on E of rank $d+1$. The volume polynomial $VP_M^{\nabla}(\underline{t}) \in \mathbb{Q}[t_F \mid F \in \mathscr{L}_M^{\geq 2}]$ of $A_{\nabla}^{\bullet}(M)$ is

$$VP_M^{\nabla}(\underline{t}) = \sum_{(F_1,\ldots,F_d)} t_{F_1}\cdots t_{F_d}$$

where the sum is over ordered collections of nonempty flats $F_1,\ldots,F_d$ of M satisfying DHR(M). Alternatively, we have

$$VP_M^{\nabla}(\underline{t}) = \sum_{\{F_1^{d_1},\ldots,F_k^{d_k}\}} \binom{d}{d_1,\ldots,d_k} t_{F_1}^{d_1}\cdots t_{F_k}^{d_k}$$

where the sum is over size d multisets $\{F_1^{d_1},\ldots,F_k^{d_k}\}$ of flats of M satisfying DHR(M).

One recovers the following central result of [Pos09] by setting $M = U_{|E|,E}$.

Theorem 5.2.6. [Pos09, Corollary 9.4] The volume polynomial $VP_{U_{n+1,n+1}}^{\nabla}(\underline{t})$ of $A_{\nabla}^{\bullet}(X_{A_n})$ is

$$VP_{U_{n+1,n+1}}^{\nabla}(\underline{t}) = \sum_{(S_1,\ldots,S_n)} t_{S_1}\cdots t_{S_n}$$

where the sum is over ordered collections of nonempty subsets $S_1,S_2,\ldots,S_n$ such that $|\bigcup_{j\in J} S_j| \geq |J|+1$ for any $\emptyset \subsetneq J \subseteq \{1,\ldots,n\}$.

The volume polynomial VP_M of the more classical presentation $A^{\bullet}(M)$ of the Chow ring of a matroid M by generators $\{x_F \mid F \in \mathscr{L}_M \setminus \{\emptyset, E\}\}$ was computed in [Eur20]. While the two polynomials VP_M and VP_M^{∇} are related by a linear change of coordinates, it is not clear at the time of writing how one formula can be derived directly from the other.

5.3 Volume polynomial of a matroid is Lorentzian

Motivated by Remark 5.1.6 and the fact that the simplicial generators of $A_{\nabla}^{\bullet}(M)$ are combinatorially nef divisors, we prove here that the volume polynomial VP_M^{∇} of the simplicial presentation $A_{\nabla}^{\bullet}(M)$ is Lorentzian.

Theorem 5.3.1. The volume polynomial $VP_M^\nabla \in \mathbb{R}[t_F \mid F \in \mathscr{L}_M^{\geq 2}]$ of a loopless matroid M is Lorentzian.

As an immediate corollary, by applying Theorem 5.1.5 to Theorem 5.3.1 we obtain:

Corollary 5.3.2. The volume polynomial VP_M^∇, as a polynomial in $\mathbb{R}[t_F \mid F \in \mathscr{L}_M^{\geq 2}]$, is strongly log-concave in the positive orthant $\mathbb{R}_{>0}^{\mathscr{L}_M^{\geq 2}}$. In other words, as a function $A^1(M) \to \mathbb{R}$, the polynomial VP_M^∇ is strongly log-concave in the interior of the cone $\mathscr{K}_M^\nabla$ generated by the simplicial generators.

We will show that the volume polynomial VP_M^∇ of a loopless matroid M satisfies the two conditions listed in Theorem 5.1.3. First, we see that the dragon Hall-Rado condition description for the support of VP_M^∇ implies that VP_M^∇ has M-convex support.

Proposition 5.3.3. Let $\{F_1, \dots, F_d\}$ and $\{G_1, \dots, G_d\}$ be two multisets of flats of M such that both $t_{F_1} \cdots t_{F_d}$ and $t_{G_1} \cdots t_{G_d}$ are in the support of VP_M^∇. If (without loss of generality) G_d is a flat which appears more times in $\{G_1, \dots, G_d\}$ than it does in $\{F_1, \dots, F_d\}$, then there exists another flat F_m which appears more times in $\{F_1, \dots, F_d\}$ than it does in $\{G_1, \dots, G_d\}$ such that $t_{F_1} \cdots t_{F_d} t_{G_d} / t_{F_m}$ is in the support of VP_M^∇.

Proof. We borrow standard language from (poly)matroid theory. Let us call a multiset of flats $\{A_1, \dots A_k\}$ *dependent* if $\mathrm{rk}_M(\bigcup_{j=1}^k A_j) \leq k$. We claim that the multiset of flats $\{F_1, \dots, F_d, G_d\}$ contains a unique minimally dependent multiset of flats X, which we call a *circuit*. The theorem will follow from this claim because the cirucuit X is not fully contained in $\{G_1, \dots, G_d\}$, hence we can let F_m be any flat in X which appears more times in $\{F_1, \dots, F_d\}$ than it does in $\{G_1, \dots, G_d\}$.

To prove the claim, suppose to the contrary that $\{R_1, \dots, R_a\}, \{S_1, \dots, S_b\}$ are two distinct circuits which are subsets of $\{F_1, \dots F_d, G_d\}$. Let $\{T_1, \dots, T_c\} = \{R_1, \dots, R_a\} \cap \{S_1, \dots, S_b\}$. We claim that $\{T_1, \dots, T_c\}$ is dependent. Suppose to the contrary that $\mathrm{rk}_M(\bigcup_{j=1}^c T_j) \geq c+1$. By assumption $\mathrm{rk}_M(\bigcup_{j=1}^a R_j) = a$ and $\mathrm{rk}_M(\bigcup_{j=1}^b S_j) = b$. Let R and S be the joins of the elements in $\{R_1, \dots, R_a\}$ and $\{S_1, \dots, S_b\}$, respectively. Submodularity gives that $\mathrm{rk}_M(R \cup S) \leq \mathrm{rk}_M(R) + \mathrm{rk}_M(S) - \mathrm{rk}_M(R \cap S) = a + b - \mathrm{rk}_M(R \cap S) \leq a + b - \mathrm{rk}_M(\bigcup_{j=1}^c T_j) \leq a + b - c - 1$. Without loss of generality, assume that $G_d = R_a = S_b = T_c$. We have that $R = \bigvee_{j=1}^{a-1} R_j = \bigvee_{j=1}^a R_j$ and $S = \bigvee_{j=1}^{b-1} S_j = \bigvee_{j=1}^b S_j$. Otherwise $\{R_1, \dots, R_{a-1}\}$ and $\{S_1, \dots, S_{b-1}\}$ would both be dependent in $\{F_1, \dots, F_d\}$. Therefore the join of the elements in $\{R_1, \dots, R_{a-1}, S_1, \dots, S_{b-1}\} \setminus \{T_1, \dots T_{c-1}\}$ is $R \cup S$ and $|\{R_1, \dots, R_{a-1}, S_1, \dots, S_{b-1}\} \setminus \{T_1, \dots T_{c-1}\}| = (a-1) + (b-1) - (c-1) = a + b - c - 1$. But, as calculated above $\mathrm{rk}_M(R \cup S) \leq a + b - c - 1$, therefore the set $\{R_1, \dots, R_{a-1}, S_1, \dots, S_{b-1}\} \setminus \{T_1, \dots T_{c-1}\}$ is dependent in $\{F_1, \dots, F_d\}$, a contradiction. $\square$

Remark 5.3.4. Suppose M has a realization $\mathscr{R}(M)$. By Remark 2.3.2.(2), the wonderful compactification $Y_{\mathscr{R}(M)}$ is embedded in the product of projective spaces

$$\prod_{F \in \mathscr{L}_M \setminus \{\emptyset\}} \mathbb{P}(V^*/L_F).$$

Our simplicial generators are pullbacks of the hyperplane classes of the projective spaces $\mathbb{P}(V^*/L_F)$ (see Remark 3.2.7). Thus, in this case, that the support of VP_M^∇ is M-convex follows from the result of [CLZ16] that the multidegree of an irreducible variety in a product of projective spaces is has M-convex support.

Proof of Theorem 5.3.1. Let M be a loopless matroid of rank $r = d+1$. There is nothing to prove if $d = 1$, so we assume $d \geq 2$. The support of VP_M^∇ is M-convex by the previous proposition. Observe that for a flat G of rank at least 2, we have

$$\frac{\partial}{\partial t_G} VP_M^\nabla(t) = d \int_M h_G \cdot \Big(\sum_{F \in \mathscr{L}_M^{\geq 2}} t_F h_F \Big)^{d-1} = d \int_{T_G(M)} \Big(\sum_{F \in \mathscr{L}_M^{\geq 2}} t_F h_{\mathrm{cl}_{T_G(M)}(F)} \Big)^{d-1}.$$

Now, suppose $\{F_1, \ldots, F_{d-2}\}$ is a multiset of size $d-2$ consisting of flats of M with rank at least 2. If $\{F_1, \ldots, F_{d-2}\}$ does not satisfy DHR(M), then $\partial_{t_{F_1}} \cdots \partial_{t_{F_{d-2}}} VP_M^\nabla \equiv 0$, so we may assume that $\{F_1, \ldots, F_{d-2}\}$ satisfies DHR(M). One computes that

$$\partial_{t_{F_1}} \cdots \partial_{t_{F_{d-2}}} VP_M^\nabla(t) = \frac{d!}{2!} \int_{M'} \Big(\sum_{F \in \mathscr{L}_M^{\geq 2}} t_F h_{\mathrm{cl}_{M'}(F)} \Big)^2$$

where $M' = M \wedge H_{F_1} \wedge \cdots \wedge H_{F_{d-2}}$ is a loopless matroid of rank 3. By Proposition 5.1.4, it suffices to check that $VP_{M'}^\nabla$ is Lorentzian. For any loopless matroid M' of rank 3, the degree 1 part $A^1_\nabla(M)$ of its Chow ring has the simplicial basis $\{h_E\} \cup \{h_F : \mathrm{rk}_{M'}(F) = 2\}$. Noting that $\int_{M'} h_E \cdot h_E = 1$, $\int_{M'} h_E \cdot h_F = 1$, and $\int_{M'} h_F \cdot h_{F'} = 1$ if $F \neq F'$ and 0, otherwise the Hessian matrix of the quadratic form $VP_{M'}^\nabla$ is two times the matrix

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix}, \text{ which reduces to } \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{bmatrix}$$

by symmetric Gaussian elimination. $\square$

Chapter 6

The Hodge theory of matroids in degrees at most one

The **reduced characteristic polynomial** of M is defined as

$$\bar{\chi}_M(t) := \frac{1}{t-1} \sum_{F \in \mathscr{L}_M} \mu(\varnothing, F) t^{\mathrm{rk}(M)-\mathrm{rk}(F)} = \sum_{k=0}^{d} (-1)^k \mu^k(M) t^{d-k}$$

where $\mu(-,-)$ is the Möbius function of the lattice $\mathscr{L}_M$ and $\mu^i(M)$ is the absolute value of the ith coefficient of $\bar{\chi}_M(t)$. The Heron-Rota-Welsh conjecture stated that

$$\mu^{k-1}(M)\mu^{k+1}(M) \leq \mu^k(M)^2 \quad \text{for } 0 < k < d.$$

In [AHK18] the authors show that the Chow ring of matroids satisfy the Kähler package, a property enjoyed by the cohomology ring of a smooth projective complex variety. A particular portion of the Kähler package, the Hodge-Riemann relation in degree 1, implies the Heron-Rota-Welsh conjecture [AHK18, §9]. This observation was among the main motivations for the development of the Hodge theory of matroids in [AHK18].

To prove that the Chow ring of a matroid satisfies the Hodge-Riemann relations, the authors of [AHK18] adapt a line of argument that originally appeared in McMullen's work on simple polytopes [McM93]. Their method employs a double induction on the rank of the matroid and on order filters in the matroid's lattice of flats: the outer induction on rank shows that the Hodge-Riemann relations hold for all ample classes if they hold for a single ample class, and the inner induction on order filters is then used to construct an ample class for which the Hodge-Riemann relations can be verified.

In this chapter, we independently establish the Hodge-Riemann relations in degree one using a similar argument. As we have established in the previous chapter that the volume polynomial VP_M^∇ of a matroid M is strongly log-concave in the subcone $\mathscr{K}_M^\nabla$ of the ample cone $\mathscr{K}_M$, we are able to avoid working with generalized Bergman fans induced by order filters and the flipping operation which interpolates between them.

Thus our proof involves only classical Bergman fans associated to matroids and takes the form of a single induction on rank alone.

6.1 The Kähler package in degree one and log-concavity

We begin by discussing here the statements of the Kähler package, and how in degree one they relate to log-concavity. We then provide some generalities on the inductive paradigm for proving Kähler package for Chow cohomology rings of fans, which was given in [AHK18] adapted from the earlier work [McM93].

Definition 6.1.1. Let $(A^\bullet, \int)$ be a Poincaré duality $\Bbbk$-algebra of dimension d with degree map $\int$. For $\ell \in A^1$ and $0 \le i \le \lfloor \frac{d}{2} \rfloor$, we define L_ℓ^k to be the Lefschetz operator

$$L_\ell^i : A^i \to A^{d-i},\ a \mapsto \ell^{d-2i} a,$$

and define Q_ℓ^i to be Hodge-Riemann symmetric bilinear form

$$Q_\ell^i : A^i \times A^i \to \Bbbk,\ (x,y) \mapsto \int xy\ell^{d-2i}.$$

We define the set of **degree** i **primitive classes of** ℓ to be $P_\ell^i := \{x \in A^k : x\ell^{d-2i+1} = 0\}$.

Definition 6.1.2. Let $(A^\bullet, \int)$ be a Poincaré duality $\mathbb{R}$-algebra of dimension d, and let $\ell \in A^1$. For $0 \le i \le \lfloor \frac{d}{2} \rfloor$, we say that $(A^\bullet, \int)$ satisfies

- HL_ℓ^i if L_ℓ^i induces an isomorphism between A^i and A^{d-i}, and
- HR_ℓ^i if the symmetric form $(-1)^i Q_\ell^i$ is positive-definite when restricted to P_ℓ^i.

Moreover, for $\mathscr{K}$ a convex cone in A^1, we say that $(A^\bullet, \int, \mathscr{K})$ satisfies the **hard Lefschetz property** ($\mathrm{HL}_\mathscr{K}^i$), resp. the **Hodge-Riemann relation** ($\mathrm{HR}_\mathscr{K}^i$), in degree i if $A^\bullet$ satisfies HL_ℓ^i, resp. HR_ℓ^i, for all $\ell \in \mathscr{K}$.

The Poincaré duality property (PD) of $(A^\bullet, \int)$ implies that the form Q_ℓ^i is non-degenerate if and only if HL_ℓ^i holds. The properties (PD), (HL), and (HR) together are called the **Kähler package** for a graded ring $A^\bullet$. We will write $\mathrm{HL}^{\le i}$ to mean hard Lefschetz property in degrees at most i, and likewise for HR. The relation between log-concavity and the Kähler package in degree ≤ 1 was realized in various contexts; for a survey we point to [Huh18a]. We will only need the following, adapted from [BH19, Proposition 5.6]. It also appeared in [AOV18, §2.3], and is a consequence of the Cauchy interlacing theorem.

Proposition 6.1.3. Let $A^\bullet$ be a Poincaré duality $\mathbb{R}$-algebra of dimension d with degree map $\int$, and $\mathscr{K}$ a convex cone in A^1. Suppose $(A^\bullet, \int, \mathscr{K})$ satisfy $\mathrm{HL}_\mathscr{K}^0$ and $\mathrm{HR}_\mathscr{K}^0$. Then the following are equivalent:

(1) The volume function $\mathrm{vol} : A^1 \to \mathbb{R}$, $\ell \mapsto \int \ell^d$ is log-concave on $\mathscr{K}$, and

(2) for any $\ell \in \mathscr{K}$, the symmetric form Q^1_ℓ has exactly one positive eigenvalue.

In particular, if the volume polynomial VP_A of $A^\bullet = \mathbb{R}[x_1, \ldots, x_s]/I$ is Lorentzian, then $(A^\bullet, \int, \mathscr{K})$ satisfies $\mathrm{HR}^{\leq 1}_{\mathscr{K}}$ where $\mathscr{K}$ is the interior of $\mathrm{Cone}(x_1, \ldots, x_s)$, provided that $A^\bullet$ satisfies $\mathrm{HL}^{\leq 1}_{\mathscr{K}}$.

We now turn to an inductive paradigm for establishing (HL) and (HR). We assume all Poincaré duality algebras to be over $\mathbb{R}$. We begin by noting an easy linear algebraic observation also made in [AHK18, Proposition 7.16].

Proposition 6.1.4. Let $(A^\bullet, \int, \mathscr{K})$ be a Poincaré duality algebra which satisfies $\mathrm{HL}^i_{\mathscr{K}}$ for $\mathscr{K}$ in a convex cone in A^1. Suppose that $(A^\bullet, \int)$ satisfies HR^i_ℓ for some $\ell \in \mathscr{K}$. Then $A^\bullet$ satisfies $\mathrm{HR}^i_{\mathscr{K}}$.

Proof. Let $\ell' \in \mathscr{K}$, and let $l(t) = t\ell + (1-t)\ell'$ for $t \in [0,1]$ be a line segment connecting ℓ and ℓ'. By convexity of $\mathscr{K}$, we know that every point on l is in $\mathscr{K}$. If the signature of the bilinear pairing $Q^i_{l(t)}$ changes along $l(t)$ starting at ℓ, then it must degenerate at some point $l(t_0)$ for $t_0 \in [0,1]$, but this violates $\mathrm{HL}^i_{\mathscr{K}}$. □

We now note how properties (HL) and (HR) behave under tensor products and transports. While these are adapted from [AHK18, §7] where they are phrased in terms of Chow cohomology rings of fans, because we restrict ourselves Kähler package up to degree 1, we can provide here easier and more direct proofs for general Poincaré duality algebras.

Proposition 6.1.5. Let $(A^\bullet, \int_A)$ and $(B^\bullet, \int_B)$ be two Poincaré duality algebras of dimension $d_A \geq 1$ and $d_B \geq 1$. Suppose that $A^\bullet$ and $B^\bullet$ satisfy $\mathrm{HR}^{\leq 1}_{\ell_A}$ and $\mathrm{HR}^{\leq 1}_{\ell_B}$, respectively, then $((A \otimes B)^\bullet, \int_{A\otimes B})$ satisfies $\mathrm{HR}^{\leq 1}_{\ell_A \otimes 1 + 1 \otimes \ell_B}$.

Before giving the proof, we remark that if $d_A = 0$ then $(A \otimes B)^\bullet \simeq B^\bullet$ (likewise if $d_B = 0$) so that the statement in the proposition is trivially satisfied after suitable modifications.

Proof. Set $\ell := \ell_A \otimes 1 + 1 \otimes \ell_B$. First, note that HR^0_ℓ follows easily from the description of the Poincaré duality algebra $(A \otimes B)^\bullet$ in Proposition 4.1.4.(1). Now, we verify that $(A \otimes B)^\bullet$ satisfies HR^1_ℓ. Let $v_1, \ldots v_m$ and $w_1, \ldots w_n$ be orthonormal bases for $P^1_{\ell_A}$ and $P^1_{\ell_B}$, respectively. Then

$$A^1 \cong \bigoplus_{i=1}^m \langle v_i \rangle \oplus \langle \ell_A \rangle \quad \text{and} \quad B^1 \cong \bigoplus_{i=1}^n \langle w_i \rangle \oplus \langle \ell_B \rangle.$$

Noting that $(A \otimes B)^\bullet$ is a Poincaré duality algebra of dimension $d = d_A + d_B$, we expand

$$\ell^{d-2} = ((\ell_A \otimes 1) + (1 \otimes \ell_B))^{d-2} = \sum_{i=0}^{d-2} \binom{d-2}{i} (\ell_A^i \otimes \ell_B^{d-i-2}).$$

The symmetric matrix for Q_ℓ^1 with respect to the above basis is given by

$$Q_\ell^1(a,b) = \begin{cases} -\binom{d-2}{d_A} & a = b = (v_i \otimes 1) \\ -\binom{d-2}{d_B} & a = b = (1 \otimes w_j) \\ \lambda\binom{d-2}{d_A-2} & a = b = (\ell_A \otimes 1) \\ \lambda\binom{d-2}{d_A} & a = b = (1 \otimes \ell_B) \\ \lambda\binom{d-2}{d_A-1} & a = (\ell_A \otimes 1) \textit{ and } b = (1 \otimes \ell_B) \\ \lambda\binom{d-2}{d_B-1} & a = (1 \otimes \ell_B) \textit{ and } b = (\ell_A \otimes 1) \\ 0 & a = (v_i \otimes 1) \textit{ and } b = (1 \otimes w_j) \textit{ or } (1 \otimes \ell_B) \\ 0 & a = (v_i \otimes 1) \textit{ or } (\ell_A \otimes 1) \textit{ and } b = (1 \otimes w_j) \end{cases}$$

where $\lambda := (\int_A \ell_A^{d_A})(\int_B \ell_B^{d_B})$.

So the matrix $Q_\ell^1(a,b)$ is a block matrix comprised of 3 blocks. By $\mathrm{HR}^1_{\ell_A}$ and $\mathrm{HR}^1_{\ell_B}$, the first two blocks are negative identity matrices induced by $\{(v_i \otimes 1)\} \times \{(v_i \otimes 1)\}$ and $\{(1 \otimes w_j)\} \times \{(1 \otimes w_j)\}$. The third and only nontrivial block is induced by $\{(\ell_A \otimes 1), (1 \otimes \ell_B)\} \times \{(\ell_A \otimes 1), (1 \otimes \ell_B)\}$, which gives the 2×2 matrix

$$M = \lambda \begin{bmatrix} \binom{d-2}{d_A-2} & \binom{d-2}{d_A-1} \\ \binom{d-2}{d_A-1} & \binom{d-2}{d_A} \end{bmatrix}.$$

One computes that $\det(M) < 0$, and hence M has signature $(+,-)$. We conclude that $Q_\ell^1(a,b)$ is nondegenerate and has exactly one positive eigenvalue completing the proof. $\square$

Proposition 6.1.6. Let $(A^\bullet = \mathbb{R}[x_1, \ldots, x_s]/I, \int)$ be a Poincaré duality algebra of dimension d, and let $\ell \in A^1$ be an effective divisor—that is, a non-negative linear combination of $\{x_1, \ldots, x_s\}$. Denote by ℓ_k the image of ℓ in $A^\bullet/\operatorname{ann}(x_k)$. For $0 \le i \le \lfloor \frac{d-1}{2} \rfloor$, if $(A^\bullet/\operatorname{ann}(x_k), \int_{x_k})$ satisfies $\mathrm{HR}^i_{\ell_k}$ for every $k = 1, \ldots, s$, then $(A^\bullet, \int)$ satisfies HL^i_ℓ.

Proof. Let $\ell = \sum_{k=1}^s c_k x_k$ with $c_k \in \mathbb{R}_{\ge 0}$, and suppose $\ell^{d-2i} f = 0$ for some $f \in A^i$. We will show that $f = 0$ necessarily. Let f_i the image of f in $A^\bullet/\operatorname{ann}(x_i)$. As $0 = \ell^{d-2i} f$, we have $0 = \ell_k^{d-2i} f_k$, and because $A^\bullet/\operatorname{ann}(x_k)$ is a Poincaré duality algebra of dimension $d-1$, we conclude that f_k belongs to the primitive space $P^i_{\ell_k}$. Now, we note that

$$0 = \int \ell^{d-2i} f^2 = \int (\textstyle\sum_k c_k x_k) \ell^{d-2i-1} f^2 = \sum_k (\int_{x_k} c_k \ell_k^{d-2i-1} f_k^2) \quad \text{and} \quad \int_{x_k} c_k \ell_k^{d-2i-1} f_k^2 \le 0 \; \forall k,$$

where the last inequalities follow from $\mathrm{HR}^i_{\ell_k}$. Moreover, as $Q^1_{\ell_k}$ is negative-definite on $P^i_{\ell_k}$, we conclude each f_k to be 0, that is, $x_k f = 0$ for all $k = 1, \ldots, s$. Since $\{x_1, \ldots, x_s\}$ generate $A^\bullet$, the Poincaré duality property of $A^\bullet$ implies that if $f \neq 0$ then there exists a polynomial $g(x)$ of degree $d - i$ such that $\int g(x) f \neq 0$, and hence we conclude that $f = 0$. □

6.2 Kähler package in degree at most one for matroids

We now specialize our discussion to the setting of matroids, and establish Kähler package in degree at most one for Chow rings of matroids. As a consequence, we recover the proof of Heron-Rota-Welsh conjecture as in [AHK18].

Theorem 6.2.1. The Chow ring of a matroid $(A^\bullet(M), \int_M, \mathscr{K}_M)$ satisfies $\mathrm{HL}^{\leq 1}_{\mathscr{K}_M}$ and $\mathrm{HR}^{\leq 1}_{\mathscr{K}_M}$.

We will prove the theorem by induction on the rank of the matroid. The key combinatorial observation that allows one to reduce the rank is the following, adapted from [AHK18, Proposition 3.5]. It underlies the well-known Hopf algebraic structure for the lattice of flats of a matroid; see [KRS99; AA17] for a detailed discussion of Hopf algebraic structures in combinatorics.

Lemma 6.2.2. Let M be a loopless matroid, and F a nonempty proper flat of M. Let ρ_F be the ray corresponding to F in the Bergman fan Σ_M of M. We have

(1) $\mathrm{star}(\Sigma_M, \rho_F) \simeq \Sigma_{M|F} \times \Sigma_{M/F}$, and consequently,

(2) an isomorphism of Poincaré duality algebras

$$A^\bullet(M)/\operatorname{ann}(x_F) \simeq (A(M|F) \otimes A(M/F))^\bullet$$

such that if $\ell \in \mathscr{K}_M$ then its image in $A^\bullet(M)/\operatorname{ann}(x_F)$ is in $(\mathscr{K}_{M|F} \otimes 1) \oplus (1 \otimes \mathscr{K}_{M/F})$.

Proof. A face of Σ_M is in $\mathrm{star}(\Sigma_M, F)$ if an only if it corresponds to a flag of flats which contains F. Any such flag naturally factors as the concatenation of two flags, one with maximal element strictly contained in F, and the other with minimal element F. This geometrically corresponds to the factorization of fans in the statement (1). For the second statement (2), first note that $M|F$ and M/F are loopless since F is a flat. Then, combine Proposition 4.1.4 and Proposition 4.1.3 with the easily verifiable fact that $A^\bullet(\Sigma \times \Sigma') \simeq (A(\Sigma) \otimes A(\Sigma'))^\bullet$ for rational fans Σ, Σ'. Lastly, (3) follows from the fact that restriction of submodular functions on lattices remain submodular under restriction to intervals in the lattice. □

The remaining key part of the induction in the proof of Theorem 6.2.1, in light of Proposition 6.1.4, is to establish HR^1_ℓ for some divisor $\ell \in \mathscr{K}_M$. In [AHK18] the authors employ the method of order filters and flips for this purpose; in our case, the Lorentzian property of the volume polynomial provides the desired key step.

Lemma 6.2.3. Let M be a loopless matroid of rank $r = d+1 \geq 2$, and recall that $\mathscr{K}_M^{\nabla}$ is the interior of the cone generated by the simplicial generators of $A^\bullet_\nabla(M)$. It is a subcone of $\mathscr{K}_M$. For any $\ell \in \mathscr{K}_M^{\nabla}$, we have $\int_M \ell^d > 0$, and when $r = d+1 \geq 3$, the form Q^1_ℓ has exactly one positive eigenvalue.

Proof. The statement $\int_M \ell^d > 0$ follows from our dragon Hall-Rado formula (Corollary 5.2.5). The second statement follows from combining Theorem 5.3.1 and Proposition 6.1.3. $\square$

Proof of Theorem 6.2.1. We proceed by induction on the rank of the matroid M. The base case consists of rank 1 matroids, for which the stated properties are trivially satisfied. Let M now be a loopless matroid of rank $r = d+1$ on a ground set E.

Observe that both properties $\mathrm{HL}^0_{\mathscr{K}_M}$ and $\mathrm{HR}^0_{\mathscr{K}_M}$ hold together if and only if $\int_M \ell^d > 0$ for all $\ell \in \mathscr{K}_M$, and that given HL^1_ℓ, the property HR^1_ℓ holds if and only if Q^1_ℓ has exactly one positive eigenvalue. Combined with Lemma 6.2.3 and Proposition 6.1.4, these facts imply that proving $\mathrm{HL}^{\leq 1}_{\mathscr{K}_M}$ is sufficient to establish $\mathrm{HR}^{\leq 1}_{\mathscr{K}_M}$. By Remark 2.2.8, any element $\ell \in \mathscr{K}_M$ can be written as a non-negative linear combination of $\{x_F \mid F \in \mathscr{L}_M \setminus \{\varnothing, E\}\}$; therefore, by Proposition 6.1.6, to establish $\mathrm{HL}^{\leq 1}_{\mathscr{K}_M}$, it suffices in turn to prove $\mathrm{HR}^{\leq 1}$ for $A^\bullet(M)/\operatorname{ann}(x_F)$ for every nonempty proper flat F. Finally, $A^\bullet(M)/\operatorname{ann}(x_F) \simeq (A(M|F) \otimes A(M/F))^\bullet$ by Lemma 6.2.2(2), so by the induction hypothesis and Proposition 6.1.5, the proof is complete. $\square$

We conclude by recounting the argument in [AHK18, §9] that the Kähler package in degree one implies the Heron-Rota-Welsh conjecture.

Lemma 6.2.4. [AHK18, Lemma 9.6] Let $\ell_1, \ell_2 \in A^1(M)$. If ℓ_2 is nef, then

$$\left(\int_M \ell_1^2 \ell_2^{d-2}\right)\left(\int_M \ell_2^2 \ell_2^{d-2}\right) \leq \left(\int_M \ell_1 \ell_2 \ell_2^{d-2}\right)^2.$$

Proof. Suppose first that ℓ_2 is ample. By Theorem 6.2.1, $A^\bullet(M)$ satisfies $\mathrm{HL}^{\leq 1}_{\ell_2}$, so we obtain a decomposition $A^1(M) \cong \langle \ell_2 \rangle \oplus P^1_{\ell_2}$ that is orthogonal with respect to the Hodge-Riemann form $Q^1_{\ell_2}$. By $\mathrm{HR}^{\leq 1}_{\ell_2}$, $Q^1_{\ell_2}$ is positive definite on $P^1_{\ell_2}$ and negative definite on $\langle \ell_2 \rangle$; therefore, the restriction of $Q^1_{\ell_2}$ to the subspace $\langle \ell_1, \ell_2 \rangle \subset A^1(M)$ is neither positive nor negative definite, so

$$\left(\int_M \ell_1^2 \ell_2^{d-2}\right)\left(\int_M \ell_2^2 \ell_2^{d-2}\right) < \left(\int_M \ell_1 \ell_2 \ell_2^{d-2}\right).$$

If ℓ_2 is merely nef rather than ample, then for any ample element ℓ, the class $\ell_2(t) := \ell_2 + t\ell$ is ample for all $t > 0$. An ample ℓ exists by Lemma 6.2.3. Now, taking a limit as $t \to 0$ in the inequality

$$\left(\int_M \ell_1^2 \ell_2(t)^{d-2}\right)\left(\int_M \ell_2(t)^2 \ell_2(t)^{d-2}\right) < \left(\int_M \ell_1 \ell_2(t) \ell_2(t)^{d-2}\right).$$

yields the desired inequality.

□

Corollary 6.2.5. For each $0 < k < d$,

$$\mu^{k-1}(M)\mu^{k+1}(M) \leq \mu^k(M)^2.$$

Proof. This proof is reproduced from [AHK18, Proposition 9.8]. We proceed by induction on $\mathrm{rk}(M)$. When $k < d-1$, the induction hypothesis applied to the truncation $T_E(M)$ implies the inequality because the absolute values of the lower coefficients of $\bar{\chi}_{T_E(M)}$ are the same as those of $\bar{\chi}_M$. Now, consider $k = d-1$. For any $i \in E$, denote $\alpha := \sum_{i \in F} x_F \in A^1(M)$ and $\beta := \sum_{i \notin F} x_F$. Both α and β are independent of the choice of i and are nef. Proposition 9.5 of [AHK18] states that $\mu^k(M) = \int_M \alpha^{d-k}\beta^k$; therefore, the desired inequality is

$$\left(\int_M \alpha^2 \beta^{d-2}\right)\left(\int_M \beta^2 \beta^{d-2}\right) \leq \left(\int_M \alpha\beta\beta^{d-2}\right)^2.$$

Since α and β are nef, this inequality holds by Lemma 7.2.13. □

Chapter 7

Geometrically distinguished divisors on matroids

In this chapter, we study the properties of some geometrically distinguished divisors on a matroid M. Due to the nature of the material in this chapter, we freely use toric geometry and materials from §2.3 on wonderful compactifications of hyperplane arrangement complements.

Throughout, let us fix $E = \{0, 1, \dots, n\}$ as before, and notate by $\binom{E}{r}$ the set of r-subsets of E for $0 \le r \le n+1$.

7.1 The permutohedral divisor of a matroid

In this section, we study a divisor class on a matroid that is obtained as the pullback of a distinguished very ample divisor on the permutohedral variety. We show that the degree of this divisor is connected to an operation in matroid theory known as Dilworth truncation.

Recall that the permutohedral variety X_{A_n} is defined as the toric variety of the braid fan Σ_{A_n}, and the braid fan Σ_{A_n} is the normal fan of the standard permutohedron (of dimension n)

$$\Pi_n := \operatorname{Conv}(w(0, 1, \dots, n) \mid \text{all permutations } w \text{ of } E) \subset \mathbb{R}^E.$$

Among the polytopes whose normal fans equal Σ_{A_n}, the standard permutohedron is a distinguished one in the following sense. The braid fan Σ_{A_n} is the Coxeter complex of type A, and it is well-known (see for example [Pos09, Proposition 2.3]) that the standard permutohedron (up to translation) is the Minkowski sum of all positive roots of the type A root system. That is, we have

$$\Pi_n = \sum_{0 \le i < j \le n} \operatorname{Conv}(\mathbf{e}_i, \mathbf{e}_j) \subset \mathbb{R}^E.$$

The standard permutohedron Π_n, by Theorem 2.1.2, defines an ample divisor class ζ_{Π_n} on Σ_{A_n}. Let us describe this divisor class ζ_{Π_n} in terms of the simplicial generators. First, we translate Π_n by $-(n+1)\mathbf{e}_E$, which yields

$$\Pi_n - (n+1)\mathbf{e}_E = -\Pi_n = \sum_{0 \leq i < j \leq n} \mathrm{Conv}(-\mathbf{e}_i, -\mathbf{e}_j) \subset \mathbb{R}^E.$$

Since the simplicial generator $h_{\{i,j\}} \in A^1(\Sigma_{A_n})$ corresponds to the standard simplex $\mathrm{Conv}(-\mathbf{e}_i, -\mathbf{e}_j)$ where $i \neq j \in E$, we conclude that

$$\zeta_{\Pi_n} = \sum_{T \in \binom{E}{2}} h_T \in A^1_{\nabla}(\Sigma_{A_n}).$$

Definition 7.1.1. We call the divisor class ζ_{Π_n} the **(standard) permutohedral divisor class on** X_{A_n}. For a loopless matroid M on E, denoting $\iota_M : \Sigma_M \hookrightarrow \Sigma_{A_n}$ for the inclusion, we define the pullback $\zeta_{\Pi_n}(M) := \iota_M^* \zeta_{\Pi_n}$ **permutohedral divisor class on** M.

Since the braid fan Σ_{A_n} is smooth, an ample divisor class is also very ample [CLS11, Theorem 6.1.15]. In other words, ζ_{Π_n} is a very ample divisor on X_{A_n} defining an embedding $|\zeta_{\Pi_n}| : X_{A_n} \hookrightarrow \mathbb{P}^{|\Pi_n \cap \mathbb{Z}^E|-1}$. If a loopless matroid M has a realization $\mathscr{R}(M)$ by $\mathbb{P}V^* \hookrightarrow \mathbb{P}^n_{\Bbbk}$, then the permutohedral divisor class on M is a very ample divisor class defining an embedding $|\zeta_{\Pi_n}(M)| : Y_{\mathscr{R}(M)} \hookrightarrow X_{A_n} \hookrightarrow \mathbb{P}^{|\Pi_n \cap \mathbb{Z}^E|-1}$ of the wonderful compactification $Y_{\mathscr{R}(M)}$. The degree of this embedding motivates the following definition for general matroids.

Definition 7.1.2. For a loopless matroid M on E, the **permutohedral volume of** M, denoted $\mathrm{PVol}(M)$, is defined to be the volume of the permutohedral divisor class, i.e.

$$\mathrm{PVol}(M) := \int_M \left(\zeta_{\Pi_n}(M)\right)^d.$$

We will compute the permutohedral volume via the Dragon Hall-Rado formula (Theorem 5.2.4). Before we do so, we need the following combinatorial notion in matroid theory called Dilworth truncation.

Definition 7.1.3. Let M be a matroid of rank $r = d+1$ on E, and write $\mathscr{L}_M^{=2}$ for the set of flats of M with rank 2. The **Dilworth truncation** of M, denoted $DT(M)$, is a new matroid whose ground set is $\mathscr{L}_M^{=2}$ and is characterized by the following property: If $G \in \mathscr{L}_M$ with $\mathrm{rk}_M(G) > 2$, then $\{F \in \mathscr{L}_M^{=2} \mid F \subseteq G\}$ is a flat of $DT(M)$.

Proposition 7.1.4. [Bry86, Theorem 7.7.5] The bases description of the Dilworth truncation $DT(M)$ of a matroid M of rank $r = d+1$ is given by

$$\mathcal{B}(DT(M)) = \{\{F_1, \ldots, F_d\} \subset \mathscr{L}_M^{=2} \mid \{F_1, \ldots, F_d\} \text{ satisfies } \mathrm{DHR}(M)\}.$$

Example 7.1.5. The Dilworth truncation of the Boolean matroid $U_{|E|,E}$ is the cyclic matroid of the complete graph on $|E|$ vertices. More generally, the Dilworth truncation $DT(U_{r,E})$ has bases consisting of forests with $r-1$ edges on $|E|$ vertices.

We refer to [Dil44; Bry85; Tum85] for more on Dilworth truncations.

In our case, we will need the following minor variant of the Dilworth truncation. For a matroid M, let us denote by $\widetilde{DT}(M)$ the matroid that has ground set $\binom{E}{2}$ and is obtained from $DT(M)$ by replacing each $F \in \mathscr{L}_M^{=2}$ with the parallel class $\{T \in \binom{E}{2} \mid \mathrm{cl}_M(T) = F\}$ and adding a loop for each $T \in \binom{E}{2}$ with $\mathrm{rk}_M(T) < 2$.

Theorem 7.1.6. For a loopless matroid M of rank $r = d+1$ on E, we have

$$\mathrm{PVol}(M) = d! \cdot |\mathcal{B}(\widetilde{DT}(M))|$$

where $\widetilde{DT}(M)$ is the minor modification of the Dilworth truncation as notated above.

Proof. We justify the computation

$$\begin{aligned}
\mathrm{PVol}(M) &= |\{\text{ordered collections } (A_1, \ldots, A_d) \text{ in } \tbinom{E}{2} \text{ satisfying } \mathrm{DHR}(M)\}| \\
&= d! \cdot |\{\text{size } d \text{ subsets } \{A_1, \ldots, A_d\} \text{ of } \tbinom{E}{2} \text{ satisfying } \mathrm{DHR}(M)\}| \\
&= d! \cdot |\mathcal{B}(\widetilde{DT}(M))|
\end{aligned}$$

as follows. The first equality follows from the Dragon Hall-Rado formula (Theorem 5.2.4). The second follows from observing that $\mathrm{DHR}(M)$ cannot be satisfied if $A_i = A_j$ for some $i \neq j$, since the rank of a two-element subset in any matroid is at most 2. The last equality then follows from the construction of $\widetilde{DT}(M)$ and the description of the bases of Dilworth truncation in Proposition 7.1.4. □

Corollary 7.1.7. Among matroids on E with rank $r = d+1$, the permutohedral volume is uniquely maximized at the uniform matroid $U_{r,E}$, with the value being

$$\mathrm{PVol}(U_{r,E}) = d! \cdot \#\{\text{forests with } d \text{ edges on vertices } E\}.$$

Proof. For the first claim, we note that in order for $\{A_1, \ldots, A_d\}$ to satisfy $\mathrm{DHR}(M)$, it must first satisfy $|\bigcup_{j\in J} A_j| \geq |J| + 1$ for all $\emptyset \subsetneq J \subseteq \{1, \ldots, d\}$, since $\mathrm{rk}_M(A) \leq |A|$ for any $A \subseteq E$. In other words, we have $\mathrm{DHR}(U_{r,E}) \supseteq \mathrm{DHR}(M)$ for any matroid M of rank r on E. If every collection $\{A_1, \ldots, A_d\} \subset \binom{E}{2}$ that satisfies $\mathrm{DHR}(U_{r,E})$ also satisfies $\mathrm{DHR}(M)$, then $M = U_{r,E}$ necessarily: In this case, every $A = \{a_1, \ldots, a_r\} \in \binom{E}{r}$ is independent in M because the collection $\{A_1 = \{a_1, a_2\}, A_2 = \{a_2, a_3\}, \ldots, A_d = \{a_{r-1}, a_r\}\}$ satisfies $\mathrm{DHR}(M)$ and hence $\mathrm{rk}_M(\bigcup_{j=1}^d A_j = A) \geq d+1 = r$. □

One recovers the classical result that the volume of a standard permutohedron equals the number of spanning trees on a complete graph by setting $r = |E|$.

An aside: Mason's conjecture for Dilworth truncations

Mason stated the following conjectures for the log-concavity of the f-vector of independent subsets of a matroid [Mas72].

Conjecture 7.1.8. Let M be a matroid of rank r on a ground set E, and let $\mathcal{I}_k(M)$ be the set of independent subsets of M of cardinality k. Then for all $0 < k < r$ we have

(a) $|\mathcal{I}_k(M)|^2 \geq |\mathcal{I}_{k-1}(M)||\mathcal{I}_{k+1}(M)|$,

(b) $|\mathcal{I}_k(M)|^2 \geq \frac{k+1}{k}|\mathcal{I}_{k-1}(M)||\mathcal{I}_{k+1}(M)|$,

(c) $|\mathcal{I}_k(M)|^2 \geq \frac{|E|-k+1}{|E|-k}\frac{k+1}{k}|\mathcal{I}_{k-1}(M)||\mathcal{I}_{k+1}(M)|$.

The implications (c)⇒(b)⇒(a) are clear. The weakest form of the conjecture (statement (a)) was proved in [AHK18, Theorem 9.9.(3)]. The strongest form (statement (c)) of the conjecture was later proved independently in [Ana+18] and [BH18]. Here, as a side note, we establish the second strongest form (statement (b)) for any restriction of a Dilworth truncation of a matroid by showing that an associated generating polynomial is a specialization of our volume polynomial. One my consider the result here as a partial progress towards [EH20, Conjecture 5.6].

Theorem 7.1.9. Conjecture 7.1.8 holds for any restriction of a Dilworth truncation of a matroid.

Proof. Let M be a loopless matroid of rank $r = d+1$ on E, and let $N := DT(M)|_S$ be the restriction of the Dilworth truncation $DT(M)$ to a subset $S \subseteq \mathcal{L}_M^{=2}$. By Theorem 5.2.4, the evaluation of the volume polynomial $VP_M^{\nabla}(t_F)$ by setting $t_F = 0$ for all $F \neq E$ and $F \notin S$ is

$$\int_M \Big(t_E h_E + \sum_{F\in S} t_F h_F\Big)^d = \sum_{k=0}^{d} \frac{d!}{(d-k)!} \sum_{I\in\mathcal{I}_k(N)} t_E^{d-k} \prod_{F\in I} t_F.$$

Up to the coefficients $\frac{d!}{(d-k)!}$, it follows from Proposition 7.1.4 that the right hand side above is the generating polynomial of the independent subsets of N, homogenized by the variable t_E. This polynomial is a nonnegative specialization of VP_M^{∇}, and is hence Lorentzian by Proposition 5.1.4 since VP_M^{∇} is Lorentzian (Theorem 5.3.1). The desired statement then follows from Theorem 5.1.5. □

7.2 The rank divisor of a matroid

Let M a be matroid on E. Its base polytope $Q(M) = \operatorname{Conv}\big(\sum_{i\in B} \mathbf{e}_i \mid B \in \mathcal{B}(M)\big) \subset \mathbb{R}^E$ can be described alternatively as

$$Q(M) = \big\{y \in \mathbb{R}^E \mid \sum_{i\in E} y_i = \mathrm{rk}_M(E) \text{ and } \sum_{i\in A} y_i \leq \mathrm{rk}_M(A) \text{ for all } A \subseteq E\big\}$$

(see [Edm70; Gel+87] for a proof). By Proposition 2.2.6 (in the form of Remark 2.2.7), the submodular property of the rank function $\mathrm{rk}_M : 2^E \to \mathbb{Z}$ implies that $Q(M)$ corresponds to the nef divisor class

$$\zeta_{Q(M)} = \sum_{\emptyset \subsetneq S \subseteq E} \mathrm{rk}_M(S) z_S \in A^1_{FY}(\Sigma_{A_n}).$$

Definition 7.2.1. For M a matroid of rank $r = d + 1$ on E, let $\zeta_{Q(M)} \in A^1(\Sigma_{A_n})$ be the nef divisor class corresponding to the base polytope $Q(M)$, which we call the **divisor class of** $Q(M)$ **on** Σ_{A_n}. If M is loopless, then the pullback $\zeta_{Q(M)}(M)$ of $\zeta_{Q(M)}$ to $A^1(M)$ is called the **rank divisor class of** M. Explicitly, we have

$$\begin{aligned} \zeta_{Q(M)} &= \sum_{\emptyset \subsetneq S \subseteq E} \mathrm{rk}_M(S) z_S \in A^1_{FY}(\Sigma_{A_n}), \quad \text{and} \\ \zeta_{Q(M)}(M) &= \sum_{F \in \mathscr{L}_M \setminus \{\emptyset\}} \mathrm{rk}_M(F) z_F \in A^1_{FY}(M). \end{aligned} \tag{7.1}$$

We define the **rank volume of** M to be[1]

$$\mathrm{RVol}(M) := \int_M \left(\zeta_{Q(M)}(M)\right)^d = \zeta^d_{Q(M)} \cap \Delta_M.$$

In this section, we study the rank divisor class of a matroid in three ways:

(a) We define the canonical divisor class and relate it to the rank divisor class.

(b) We give an expression for the rank divisor class in terms of the simplicial generators, yielding a formula for the rank volume.

(c) We investigate the extremal values of the rank volume as the matroid varies. For the maxima, we answer positively a conjecture from [Eur20]; for the minima, we state and make a partial progress towards a conjecture.

The canonical divisor class and the rank divisor class

Definition 7.2.2. For M a loopless matroid of rank r on E, we define the **canonical divisor class** of M, denoted K_M, to be

$$K_M := rz_E + \sum_{\emptyset \subsetneq F \subsetneq E} \left(\mathrm{rk}_M(F) - 1\right) z_F \in A^\bullet_{FY}(M).$$

The following remark motivates our definition of the canonical divisor of a matroid.

[1] We caution that the rank volume of M is not the same as the volume of the base polytope $Q(M)$.

Remark 7.2.3. If $\pi : \mathrm{Bl}_Y X \to X$ is the blow-up of a smooth projective variety X along a smooth subvariety $Y \subset X$, then we have the following formula for the canonical divisors [Ful98, §15.4.3]:

$$K_{\mathrm{Bl}_Y X} = \pi^* K_X + (\mathrm{codim}_X Y - 1)E_Y. \tag{7.2}$$

Now, let M be a loopless matroid of rank r on E, with a realization $\mathscr{R}(M)$ and $\mathbb{P}V^* \hookrightarrow \mathbb{P}^n$. Recall from §2.3 that the wonderful compactification $Y_{\mathscr{R}(M)}$ is constructed as a series of blow-ups from $\mathbb{P}V^*$. Moreover, divisor class $-z_E \in A^\bullet_{FY}(M) \simeq A^\bullet_{FY}(Y_{\mathscr{R}(M)})$ is the hyperplane class pullback from $\mathbb{P}V^*$. Hence, by the formula above in Equation (7.2), the canonical divisor of the wonderful compactification is

$$K_{Y_{\mathscr{R}(M)}} = rz_E + \sum_{\emptyset \subsetneq F \subsetneq E} \big(\mathrm{rk}_M(F) - 1\big) z_F.$$

Let us now relate the rank divisor class of a matroid to canonical divisor classes. Algebraically, the following proposition is almost a triviality; its geometric significance is contained in the two remarks that follow.

Proposition 7.2.4. Let us write K_{A_n} for the canonical divisor class $K_{U_{|E|,E}}$ of the Boolean matroid $U_{|E|,E}$. Then

$$K_{A_n} = \sum_{\emptyset \subsetneq S \subsetneq E} -x_S \in A^\bullet(\Sigma_{A_n}) \quad (\text{recall } A^\bullet(\Sigma_{A_n}) = A^\bullet(U_{|E|,E})).$$

Hence, for a loopless matroid M on E, denoting $K_{A_n}(M)$ to be the image of K_{A_n} under the pullback map $A^\bullet(\Sigma_{A_n}) \to A^\bullet(M)$, we have

$$K_M = \zeta_{Q(M)}(M) + K_{A_n}(M).$$

Proof. For the Boolean matroid $U_{|E|,E}$, we have

$$K_{U_{|E|,E}} = |E| z_E + \sum_{\emptyset \subsetneq S \subsetneq E} \big(|S| - 1\big) z_S = \sum_{\emptyset \subsetneq S \subseteq E} |S| z_S + \sum_{\emptyset \subsetneq S \subsetneq E} -z_S,$$

and

$$\sum_{\emptyset \subsetneq S \subseteq E} |S| z_S = \sum_{i \in E} \Big(\sum_{S \ni i} z_S \Big) = \sum_{i \in E} 0 = 0$$

by the linear relations defining $A^\bullet(\Sigma_{A_n})$, and hence,

$$K_{A_n} := K_{U_{|E|,E}} = \sum_{\emptyset \subsetneq S \subsetneq E} -x_S \in A^\bullet(\Sigma_{A_n}), \quad \text{and} \quad K_{A_n}(M) = \sum_{\emptyset \subsetneq F \subsetneq E} -x_F \in A^\bullet(M).$$

The second statement now follows from the definition of $\zeta_{Q(M)}(M)$ and K_M. $\square$

The first of two geometric contents of Proposition 7.2.4 is the following remark.

Remark 7.2.5. Suppose Σ is a rational fan. Then the canonical divisor class of the toric variety X_Σ is $K_{X_\Sigma} = -\sum_{\rho\in\Sigma(1)} x_\rho \in A^\bullet(\Sigma) \simeq A^\bullet(X_\Sigma)$ [CLS11, Theorem 8.2.3].

When the matroid M is the Boolean matroid $U_{|E|,E}$, the wonderful compactification of a realization of M is isomorphic to the permutohedral variety X_{A_n}. As X_{A_n} is the toric variety of the braid fan Σ_{A_n}, its canonical divisor class is $K_{X_{A_n}} = \sum_{\emptyset\subsetneq S\subsetneq E} -x_S \in A^\bullet(\Sigma_{A_n})$, which agrees with our $K_{U_{|E|,E}}$ by the first part of Proposition 7.2.4.

Now, suppose a loopless matroid M is not the Boolean matroid, and has a realization $\mathscr{R}(M)$. Despite the Chow equivalence $A^\bullet(Y_{\mathscr{R}(M)}) \simeq A^\bullet(X_{\Sigma_M})$ between the wonderful compactification $Y_{\mathscr{R}(M)}$ and the toric variety X_{Σ_M} of the Bergman fan Σ_M, their canonical divisor classes are evidently different: The canonical divisor class of X_{Σ_M} is $K_{\Sigma_M} = \sum_{\emptyset\subsetneq F\subsetneq E} -x_F = K_{A_n}(M) \in A^\bullet(M)$, and hence the second part of Proposition 7.2.4 implies that the difference between the canonical divisor class of $Y_{\mathscr{R}(M)}$ and X_{Σ_M} is exactly the rank divisor class $\zeta_{Q(M)}(M)$.

The following remark uses Proposition 7.2.4 to interpret the rank divisor class as the first Chern class of a normal bundle.

Remark 7.2.6. Let M be a loopless matroid on E with a realization $\mathscr{R}(M)$. Consider the conormal sequence of the embedding $\iota_M : Y_{\mathscr{R}(M)} \hookrightarrow X_{A_n}$

$$0 \to \mathscr{N}^\vee \to \iota_M^*\Omega_{X_{A_n}} \to \Omega_{Y_{\mathscr{R}(M)}} \to 0,$$

from which we see that the first Chern class $c_1(\mathscr{N})$ of the normal bundle $\mathscr{N}$ of $Y_{\mathscr{R}(M)} \subset X_{A_n}$ satisfies $-c_1(\mathscr{N}) + K_M = K_{A_n}(M)$. Hence, Proposition 7.2.4 implies that

$$\zeta_{Q(M)}(M) = c_1(\mathscr{N}).$$

In fact, we can express the Chern roots of $\mathscr{N}$ in the following way. Suppose M has the Higgs factorization

$$M^c \overset{\mathcal{K}_c}{\twoheadleftarrow} \cdots \overset{\mathcal{K}_2}{\twoheadleftarrow} M^1 \overset{\mathcal{K}_1}{\twoheadleftarrow} M^0 = U_{|E|,E}$$

where $\mathcal{K}_i$'s are the modular cuts of the elementary quotients and $\mathscr{F}_i$ the corresponding modular filters. (For an elementary quotient $M' \overset{\mathcal{K}}{\twoheadleftarrow} M$, its **modular filter** is $\mathscr{F} := \{S \subseteq E \mid \mathrm{rk}_{M'}(S) = \mathrm{rk}_M(S) - 1\}$, and we have $\mathcal{K} = \mathscr{F} \cap \mathscr{L}_M$ and $\mathscr{F} = \{S \subseteq E \mid \mathrm{cl}_{M'}(S) \in \mathcal{K}\}$.) As before, let $\alpha \in A^1(X_{A_n})$ be the hyperplane class pullback, and let us denote $\alpha_{\mathscr{F}_i} := \sum_{S\in\mathscr{F}_i} x_S \in A^1(X_{A_n})$. Suppose all the elementary quotients in the Higgs factorization are realizable, so that we have a filtration of the inclusion $Y_{\mathscr{R}(M)} \subset X_{A_n}$ as

$$Y_{\mathscr{R}(M)} = Y_{\mathscr{R}(M^c)} \subset \cdots \subset Y_{\mathscr{R}(M^1)} \subset Y_{\mathscr{R}(M^0)} = X_{A_n}.$$

At each step, one can show that $[Y_{\mathscr{R}(M^i)}] = (\alpha - \alpha_{\mathscr{F}_i}) \cdot [Y_{\mathscr{R}(M^{i-1})}]$ in $A^\bullet(X_{A_n})$. Hence, the images of $\{\alpha - \alpha_{\mathscr{F}_i}\}_{i=1,\dots,c}$ under the pullback $A^\bullet(X_{A_n}) \to A^\bullet(M)$ are the Chern roots of the normal bundle of the embedding $Y_{\mathscr{R}(M)} \subset X_{A_n}$. It is interesting to note that

$$\zeta_{Q(M)} = \sum_{i=1}^{c}(\alpha - \alpha_{\mathscr{F}_i}) \quad \text{while} \quad \Delta_M = \prod_{i=1}^{c}(\alpha - \alpha_{\mathscr{F}_i}) \quad \text{as elements in } A^\bullet(X_{A_n}).$$

Rank divisor classes in terms of the simplicial generators

We now turn to expressing $\zeta_{Q(M)}$ and $\zeta_{Q(M)}(M)$ in terms of the simplicial generators. Our computation here closely mirrors that of [ABD10]. Let us prepare by reviewing the beta invariant of a matroid, studied in [Cra67].

Definition 7.2.7. The **beta invariant** of a matroid M of rank r on E is defined as

$$\beta(M) := (-1)^r \sum_{A \subseteq E} (-1)^{|A|} \operatorname{rk}_M(A).$$

Some properties of the beta invariant follow. For proofs see [Cra67].

Theorem 7.2.8. Let M be a matroid of rank r on E.

(1) $\beta(M) = (-1)^r \sum_{F \in \mathscr{L}_M} \mu(\emptyset, F) \operatorname{rk}_M(F) = (-1)^{r-1} \left(\frac{d}{dt} \chi_M(t) \Big|_{t=1} \right) = (-1)^{r-1} \overline{\chi}_M(1).$

(2) $\beta(M) \geq 0$ and equals 0 if and only if M is disconnected or is a loop.

Let us define the **signed beta invariant** of a matroid M of rank r to be

$$\widetilde{\beta}(M) = (-1)^{r-1} \beta(M),$$

which is also equal to $\overline{\chi}_M(1)$, the sum of coefficients of the reduced characteristic polynomial of M, by Theorem 7.2.8.(1). We can express the rank divisor class by the signed beta invariants in the following way.

Proposition 7.2.9. Let M be a loopless matroid of rank r on E. Then one has

$$\zeta_{Q(M)} = \sum_{\emptyset \subsetneq S \subseteq E} (-1)^{|S|} \widetilde{\beta}(M|_S) h_S, \text{ and thus } \zeta_{Q(M)}(M) = \sum_{F \in \mathscr{L}_M^{\geq 2}} \Big(\sum_{\operatorname{cl}_M(S) = F} (-1)^{|S|} \widetilde{\beta}(M|_S) \Big) h_F.$$

We also have $\zeta_{Q(M)}(M) = \sum_{F \in \mathscr{L}_M^{\geq 2}} \Big(-\sum_{\emptyset \subseteq G \subseteq F} \mu(G, F) \operatorname{rk}_M(G) \Big) h_F$.

Proof. We first recall the definition $h_F := \sum_{G \supseteq F} -z_F$. One computes that

$$\text{if} \quad \sum_{F \in \mathscr{L}_M \setminus \{\emptyset\}} a_F h_F = \sum_{F \in \mathscr{L}_M \setminus \{\emptyset\}} b_F z_F, \quad \text{then}$$
$$\sum_{\emptyset \subsetneq G \subseteq F} -a_G = b_F, \quad \text{equivalently by Möbius inversion,} \quad a_F = \sum_{\emptyset \subsetneq G \subseteq F} -\mu(G, F) b_G. \quad (\dagger)$$

We now compute:

$$\begin{aligned}\zeta_{Q(M)} &= \sum_S \mathrm{rk}_M(S) z_S \\ &= \sum_S \Big(-\sum_{\emptyset \subseteq T \subseteq S} (-1)^{|S\setminus T|} \,\mathrm{rk}_M(T)\Big) h_S \\ &= \sum_S \Big((-1)^{|S|+\mathrm{rk}(M|_S)+1} \beta(M|_S)\Big) h_S \\ &= \sum_S (-1)^{|S|} \widetilde{\beta}(M|_S) h_S, \text{ and,}\end{aligned}$$

$$\begin{aligned}\zeta_{Q(M)}(M) &= \sum_F \mathrm{rk}_M(F) z_F \\ &= \sum_F \Big(-\sum_{\emptyset \subseteq G \subseteq F} \mu(G,F)\,\mathrm{rk}_M(G)\Big) h_F.\end{aligned}$$

In both cases, the first equality is Equation (7.1); the second equality is our observation (†), where $\emptyset$ is included in the summation because $\mathrm{rk}_M(\emptyset) = 0$; and the third follows from either the definition of β. The first formula for $\zeta_{Q(M)}(M)$ follows from the one for $\zeta_{Q(M)}$ since $h_S(M) = h_{\mathrm{cl}_M(S)}(M)$. □

A formula for the rank volume immediately follows by applying Theorem 5.2.4 to the above proposition, and is further simplified by applying Theorem 7.2.8.(2).

Corollary 7.2.10. Let M be a loopless matroid of rank $r = d+1$ on E. Then

$$\mathrm{RVol}(M) = \sum_{(F_1,\dots,F_d)} \prod_{i=1}^{d} \Big(\sum_{\mathrm{cl}_M(S)=F_i} (-1)^{|S|} \widetilde{\beta}(M|_S)\Big)$$

where the summation is over all ordered sequences of flats $(F_1, \dots, F_d)$ that satisfies $\mathrm{DHR}(M)$ and $M|_{F_i}$ is a connected matroid for each $1 \le i \le d$.

Extrema of rank volumes

We now consider the extremal values of the rank volume $\mathrm{RVol}(M)$ as M ranges over all loopless matroids with fixed rank and ground set, and ask when those extrema are achieved. Despite a formula for the rank volume (Corollary 7.2.10), the presence of signs in the formula makes it difficult to be applied to the questions here. Some of our arguments here are inherently geometric, and formulating their combinatorial counterparts remains open.

We start with the maximum.

Theorem 7.2.11. Among all loopless matroids of rank $r = d+1$ on E, the rank volume is maximized at the uniform matroid $U_{r,E}$ with the value $\mathrm{RVol}(U_{r,E}) = c^d$ where $c = |E| - r$.

A similar statement appeared in [Eur20] for the "shifted rank volumes" of matroids. The shifted rank volume of a loopless matroid M of rank r is $\int_M \left(\zeta_{Q(M)}(M) + r\alpha(M)\right)^{r-1}$. The author of [Eur20] showed that shifted rank volumes are maximized at uniform matroids among realizable matroids, and conjectured that the same statement holds without the realizability condition. The proof of Theorem 7.2.11 here carries over with essentially no change to the shifted rank volumes, and hence resolves this conjecture from [Eur20]. The conjecture was partially motivated by the search for analogues of Newton-Okounkov bodies for matroids; see [Eur20, Remark 1.2].

We prepare for the proof of Theorem 7.2.11 with the following manipulation.

Lemma 7.2.12. Let M be a matroid of rank r on E, with corank $c := |E| - r$. Recall the notation $\alpha = -z_E \in A^1(\Sigma_{A_n})$ as the hyperplane class. Then

$$\zeta_{Q(M)} = c\alpha - E, \text{ where } E = \sum_{\emptyset \subsetneq S \subsetneq E} \big(|S| - \mathrm{rk}_M(S)\big)x_S \in A^1(\Sigma_{A_n}) \text{ is an effective divisor.}$$

Proof. Let us start by noting that $\sum_{S \ni i} z_S = 0$ for any $i \in E$ by the definition of $A^\bullet(\Sigma_{A_n})$, and hence, we have $\sum_{\emptyset \subsetneq S \subseteq E} |S| z_S = \sum_{i \in E} \left(\sum_{S \ni i} z_S\right) = \sum_{i \in E} 0 = 0$. We now compute

$$\begin{aligned} \zeta_{Q(M)} &= \sum_{\emptyset \subsetneq S \subseteq E} \mathrm{rk}_M(S) z_S \\ &= \sum_S (\mathrm{rk}_M(S) - |S|) z_S + \sum_S |S| z_S \\ &= c\alpha - \sum_{\emptyset \subsetneq S \subsetneq E} (|S| - \mathrm{rk}_M(S)) x_S. \end{aligned}$$

That $|S| - \mathrm{rk}_M(S) \geq 0$ for all $S \subseteq E$ is a defining property of rank functions of matroids. □

We will use this lemma in conjunction with the following geometric observation.

Lemma 7.2.13. Let D_1 and D_2 be nef divisors on a smooth projective variety X of dimension d such that $D_1 - D_2$ is effective. Then we have $\int_X D_1^d \geq \int_X D_2^d$.

Proof. Since D_1 is nef, we have

$$\int_X D_1^d = \lim_{m \to \infty} \frac{h^0(\mathscr{O}_X(mD_1))}{m^d/d!},$$

and likewise for D_2 [Laz04, Corollary 1.4.41]. The desired inequality thus follows from the fact that $D_1 - D_2$ effective implies $H^0(\mathscr{O}_X(mD_1)) \supseteq H^0(\mathscr{O}_X(mD_2))$ for all $m \geq 0$. □

Proof of Theorem 7.2.11. From Lemma 7.2.12, we first note that $\zeta_{Q(U_{r,E})}(U_{r,E}) = c\alpha(U_{r,E})$. Recall that the hyperplane class $\alpha(U_{r,E}) = -z_E(U_{r,E}) = h_E(U_{r,E}) \in A^1_\nabla(U_{r,E})$, so that by Theorem 5.2.4

$$\mathrm{RVol}(U_{r,E}) = c^d \int_M h_E(U_{r,E})^d = c^d.$$

Now, let M be a loopless matroid of rank r on E. Let us denote $\zeta := c\alpha \in A^1(X_{A_n})$. Since $\zeta^d \cap \Delta_M = \int_M (c\alpha(M))^d = c^d \int_M h_E(M)^d = c^d$, we are done once we show

$$\zeta_{Q(M)}^d \cap \Delta_M \leq \zeta^d \cap \Delta_M.$$

To finish the proof by using Lemma 7.2.13, we first use [Huh14, Corollary 34], which states that the Bergman class Δ_M, considered an element of the Chow ring $A^\bullet(X_{A_n})$, is effective. That is, letting $\delta : A^\bullet(X_{A_n}) \xrightarrow{\sim} \mathrm{MW}_{n-\bullet}(\Sigma_{A_n})$ be the isomorphism from Theorem 2.1.6, one can write

$$\delta^{-1}\Delta_M = \sum_{\sigma \in \Sigma_{A_n}(c)} a_\sigma [V(\sigma)] \in A^c(X_{A_n}) \quad \text{with } a_\sigma \geq 0 \text{ for all } \sigma,$$

where $V(\sigma)$ is the torus orbit closure in X_{A_n} corresponding to $\sigma \in \Sigma_{A_n}(c)$. Thus, for any divisor class $\xi \in A^1(X_{A_n})$, we have

$$\xi^d \cap \Delta_M = \int_{X_{A_n}} \sum_{\sigma \in \Sigma_{A_n}(c)} \xi^d \cdot a_\sigma [V(\sigma)] = \sum_{\sigma \in \Sigma_{A_n}(c)} a_\sigma \int_{V(\sigma)} (\xi|_{V(\sigma)})^d$$

The theorem now follows from Lemma 7.2.13 since ζ and $\zeta_{Q(M)}$ are nef divisor classes on X_{A_n} whose difference is effective by Lemma 7.2.12, and the same is true for their pullbacks to any torus orbit closure $V(\sigma)$. □

We now consider the minimum. Since $\zeta_{Q(M)}(M)$ is combinatorially nef, the Hodge-Riemman relations in degree 0 (Theorem 6.2.1) implies that $\mathrm{RVol}(M) \geq 0$. We have the following conjecture for when 0 is attained.

Conjecture 7.2.14. For a loopless matroid M, one has $\mathrm{RVol}(M) = 0$ if and only if M is disconnected.

One direction follows from the fact that base polytopes of disconnected matroids are not full dimensional. Recall that the dimension of the base polytope $Q(M)$ of a matroid M on E is $|E| - \mathrm{comp}(M)$, where $\mathrm{comp}(M)$ is the number of components of M (see [FS05, Proposition 2.4] for a proof). In particular, the base polytope $Q(M)$ is full dimensional in the (affine translate of) the dual space $N_\mathbb{R}^\vee$ of $N_\mathbb{R} = \mathbb{R}^E/\mathbb{R}\mathbf{e}_E$ if and only if M is connected.

Proposition 7.2.15. If M is disconnected, then $\mathrm{RVol}(M) = 0$.

Proof. Let $\Delta_{Q(M)} \in \mathrm{MW}^1(\Sigma_{A_n})$ be the Minkowski weight of codimension 1 corresponding to the divisor $\zeta_{Q(M)}$. By Proposition 2.1.7, it is the 1-codimemsional skeleton of the normal fan $\Sigma_{Q(M)}$ with all weights equal to 1, since the edges of $Q(M)$ all have lattice length 1. We have $\mathrm{RVol}(M) = \zeta_{Q(M)}^d \cap \Delta_M = \Delta_{Q(M)}^d \cdot \Delta_M$, where in the last expression the multiplication is the stable intersection of Minkowski weights. If M is disconnected, then the base polytope $Q(M)$ has dimension $< n$, and thus Lemma 2.2.3 implies that $\Delta_{Q(M)}$ and Δ_M share a nontrivial lineality space, so that if the stable intersection $\Delta^d_{Q(M)} \cdot \Delta_M$ is nonzero, then it must also have this positive dimensional lineality space. But $\Delta^d_{Q(M)} \cdot \Delta_M$ has to be a zero-dimensional Minkowski weight. $\square$

We provide a proof for the converse when M is realizable.

Proposition 7.2.16. If M is realizable and connected, then $\mathrm{RVol}(M) > 0$.

Proof. Let M be a connected realizable matroid of rank $r = d + 1$ on E with a realization $\mathscr{R}(M)$ by $\mathbb{P}V^* \hookrightarrow \mathbb{P}^n$. Let $\mathring{Y}_{\mathscr{R}(M)}$ be the hyperplane arrangement complement, which is a subvariety of the torus $T_N = (\Bbbk^*)^{n+1}/\Bbbk^*$. Let $X_{Q(M)}$ be the toric variety of the lattice polytope $Q(M)$. Its torus is T_N because M connected implies that $Q(M)$ is full-dimensional. (If M is disconnected, then $Q(M)$ is not full-dimensional, so that the torus of the toric variety $X_{Q(M)}$ is a nontrivial quotient of T_N).

As the normal fan of $Q(M)$ coarsens Σ_{A_n}, we have a map $X_{A_n} \to X_{Q(M)}$, and the distinguished very ample divisor $D_{Q(M)}$ on $X_{Q(M)}$ corresponding to the polytope $Q(M)$ pulls back to the divisor class $\zeta_{Q(M)}$ on X_{A_n}. Thus, the rank divisor class $\zeta_{Q(M)}(M)$ on the wonderful compactification $Y_{\mathscr{R}(M)}$ is a base-point-free divisor class defining the map $\varphi : Y_{\mathscr{R}(M)} \hookrightarrow X_{A_n} \to X_{Q(M)} \hookrightarrow \mathbb{P}^{|Q(M)\cap\mathbb{Z}^E|-1}$. Let Y_φ be the image of this map. The variety Y_φ has dimension d because the map $X_{A_n} \to X_{Q(M)}$ is identity on the torus T_N and so Y_φ contains $\mathring{Y}_{\mathscr{R}(M)}$. Thus, the degree of Y_φ as a subvariety of $\mathbb{P}^{|Q(M)\cap\mathbb{Z}^E|-1}$ is given by $\int_{Y_{\mathscr{R}(M)}} \zeta_{Q(M)}(M)^d = \mathrm{RVol}(M)$, and hence the rank volume is positive. $\square$

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Appendix A

A brief tour of matroids

We review the fundamentals of matroid theory in this appendix. For a detailed treatment of matroids, along with proofs of statements here, we point to [Wel76], [Whi86], or [Oxl11]. Geometrically oriented readers may also enjoy [Kat16], [Bak18], and [Huh18b].

Notation. As it is customary in matroid theory, for a subset $S \subseteq E$ and $i \in E$ we write $S \cup i$ for $S \cup \{i\}$ and write $S \setminus i$ for $S \setminus \{i\}$.

A.1 Linear subspaces and matroids

Matroids admit several equivalent characterizations. Here we review characterizations of matroids by (i) bases, (ii) rank functions, (iii) flats, and (iv) base polytopes.

Definition A.1.1 (Bases)**.** Let E be a finite set and $0 \leq r \leq |E|$. A **matroid** M of **rank** r on the **ground set** E is the data of $M = (E, \mathcal{B}(M))$, where $\mathcal{B}(M) \subseteq \binom{E}{r}$ is a collection of r-subsets of E satisfying

(B1) $\mathcal{B}(M) \neq \emptyset$, and

(B2) for any $B, B' \in \mathcal{B}(M)$ and $x \in B \setminus B'$, there exists $y \in B' \setminus B$ such that $B - x \cup y \in \mathcal{B}(M)$.

The collection $\mathcal{B}(M)$ is called the set of **bases** of M.

Example A.1.2 (Realizable matroids)**.** Let $E = \{v_0, \ldots, v_n\}$ be a set of $n+1$ vectors spanning a $\Bbbk$-vector space V of rank r. The subsets of E that are bases of V form a basis of a matroid of rank r on ground set E.

In other words, given a surjection $v : \Bbbk^E \twoheadrightarrow V$ where $\mathbf{e}_i \mapsto v_i$ for $i \in E$, or equivalently an r-dimensional linear subspace $V^* \subseteq \Bbbk^E$, we denote $M(V^*)$ to be the matroid whose ground set is identified with the image of the standard basis under $\Bbbk^E \twoheadrightarrow V$. In this way, matroids of rank r on a ground set E are combinatorial models of r-dimensional linear subspaces in $\Bbbk^E$. Matroids that arise in this way are called **realizable matroids**. A **realization** of a matroid M is the data of $\Bbbk^E \twoheadrightarrow V$ (equivalently $V^* \hookrightarrow \Bbbk^E$) such that $M(V^*) = M$.

Example A.1.3. Let $G = (V(G), E(G))$ is a finite connected graph with the incidence matrix I_G. The columns of I_G define a matroid of rank $|V(G)| - 1$ on the ground set $E(G)$. The bases of this matroid are the spanning trees of G. Matroids that arise in this way are called **graphical matroids**.

The graphical example motivates the following terminologies. For a matroid M on a ground set E, an element $e \in E$ is a **loop** if it not contained in any basis of M, and e is a **coloop** if it is contained in every basis of M.

Example A.1.4. For $0 \leq r \leq |E|$, the **uniform matroid** of rank r is a matroid $U_{r,E}$ whose bases are all r-subsets of E. When $r = |E|$, the uniform matroid $U_{r,E}$ is also called the Boolean matroid on E.

The **rank function** $\mathrm{rk}_M : 2^E \to \mathbb{Z}$ of a matroid M on E is defined by

$$\mathrm{rk}_M(S) = \max\{|S \cap B| \colon B \in \mathcal{B}(M)\} \quad \text{for all subsets } S \text{ of the ground set.}$$

It combinatorially models dimensions of linear subspaces, because if M has a realization $v : \Bbbk^E \twoheadrightarrow V$ then rk_M satisfies $\mathrm{rk}_M(S) = \dim_{\Bbbk}(\mathrm{span}_{\Bbbk}\{v_i \mid i \in S\})$. Matroids can be characterized in terms of rank functions in the following way.

Proposition A.1.5. A function $\mathrm{rk} : 2^E \to \mathbb{Z}$ on a finite set E is a rank function of a matroid on E if and only if

(R1) $0 \leq \mathrm{rk}(S) \leq |S|$ for any $S \subseteq E$,

(R2) $\mathrm{rk}(S_1) \leq \mathrm{rk}(S_2)$ for any $S_1 \subseteq S_2 \subseteq E$, and

(R3) $\mathrm{rk}(S_1 \cup S_2) + \mathrm{rk}(S_1 \cap S_2) \leq \mathrm{rk}(S_1) + \mathrm{rk}(S_2)$ for any $S_1 \subseteq E$, $S_2 \subseteq E$.

A subset $F \subseteq E$ is a **flat** of a matroid M if $\mathrm{rk}_M(F \cup x) > \mathrm{rk}_M(F)$ for all $x \in E \setminus F$. The set of all flats of M is denoted $\mathscr{L}_M$. With respect to inclusion, the set $\mathscr{L}_M$ is a poset that is a geometric lattice. The atoms of this lattice, that is, flats F of M with $\mathrm{rk}_M(F) = 1$, are called the **atoms** of M. The set of atoms of M is denoted $\mathfrak{A}(M)$.

When M has a realization $v : \Bbbk^E \twoheadrightarrow V$, the flats of M correspond to linear subspaces of V obtained as spans of subsets the vectors $\{v_i \mid i \in E\} \subset V$. Dually, a flat F of M with $\mathrm{rk}_M(F) = c$ corresponds to a c-codimensional linear subspace $L_F \subseteq V^*$ by

$$L_F = \{f \in V^* \mid f(v_i) = 0 \ \forall i \in F\}.$$

In particular, an atom $a \in (M)$ corresponds to the hyperplanes L_a in V^*, and we thus have a hyperplane arrangement $\{L_a\}_{a \in \mathfrak{A}(M)}$ on V^*. Often one projectivizes this, and consideres the hyperplane arrangement $\{\mathbb{P}L_a\}_{a \in \mathfrak{A}(M)}$ on $\mathbb{P}V^*$.

Matroids can be characterized in terms of flats in the following way.

Proposition A.1.6. Let $\mathscr{L}$ be a collection of subsets of a finite set E. Then $\mathscr{L}$ is a set of flats of a matroid if and only if

(F1) the ground set E is in $\mathscr{L}$,

(F2) if $F_1, F_2 \in \mathscr{L}$ then $F_1 \cap F_2 \in \mathscr{L}$, and

(F3) for each $F \in \mathscr{L}$, let $\mathscr{L}^{\gtrdot F}$ be the set of flats that cover F, i.e.

$$\mathscr{L}^{\gtrdot F} := \{G \in \mathscr{L} \mid G \supsetneq F, \text{ and } G \supsetneq G' \supsetneq F \Rightarrow G' \notin \mathscr{L}\}.$$

Then for every $F \in \mathscr{L}$, the collection $\{G \setminus F \mid G \in \mathscr{L}^{\gtrdot F}\}$ partitions $E \setminus F$.

Example A.1.7. Let us consider the uniform matroid $U_{3,4}$ of rank 3 on a ground set $\{0, 1, 2, 3\}$. The matroid $U_{3,4}$ can be realized as four general vectors in $\Bbbk^3$. For instance, it is realized as the columns of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

The lattice of flats of M and the (projective) hyperplane arrangement (of a realization of M) are drawn below.

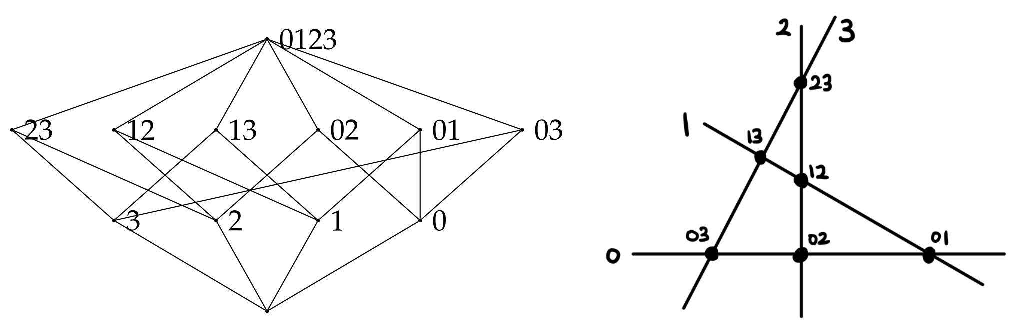

Figure A.1: The lattice of flats and the hyperplane arrangement in Example A.1.7

Example A.1.8. Let us consider the matroid on M on a ground set $\{0, 1, 2, 3, 4\}$ of rank 4 with bases $\{0123, 0124, 0134\}$. The matroid M can be realized as five vectors in $\Bbbk^4$ where three vectors v_2, v_3, v_4 are in a common plane and two other vectors v_0, v_1 are in general position. For instance, it is realized as the columns of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

The lattice of flats of M and the (projective) hyperplane arrangement (of a realization of M) are drawn below.

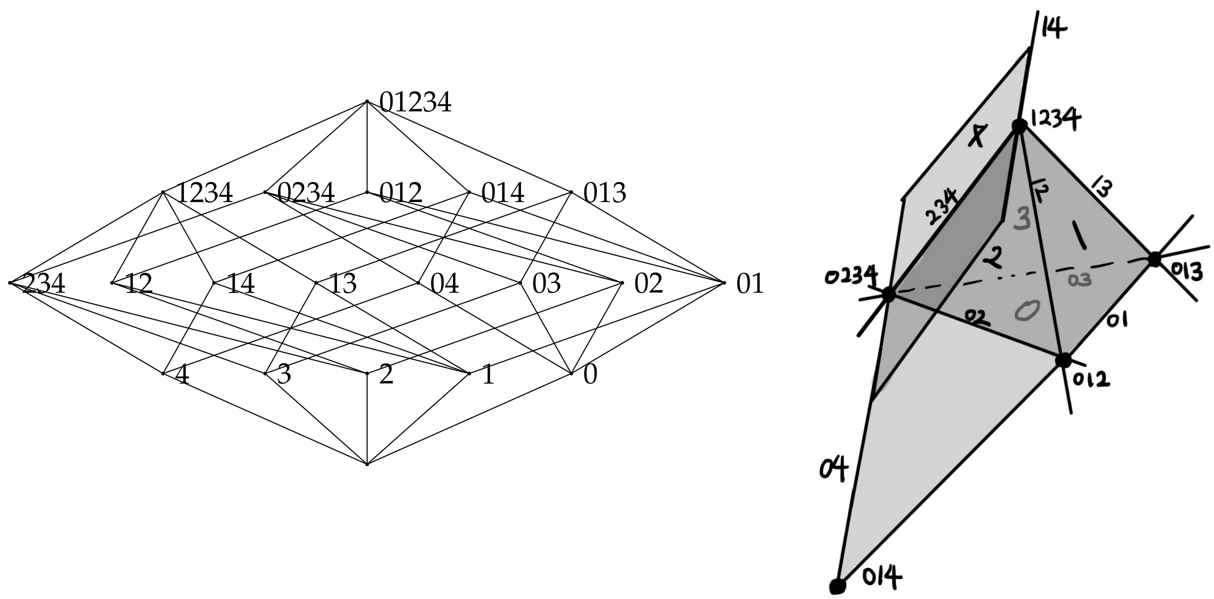

Figure A.2: The lattice of flats and the hyperplane arrangement in Example A.1.7

The study of Chow rings of matroids is motivated by certain compactifications of hyperplane arrangement complements. Hence, it heavily utilizes the properties of the flats of a matroid, as can be seen in [FY04], [AHK18], and this thesis. We now mention another route through which matroid theory interacts with algebraic geometry.

The **base polytope** $\mathcal{Q}(M)$ of a matroid M on E is the polytope

$$\mathrm{Conv}\left(\sum_{i\in B}\mathbf{e}_i \colon B\in\mathcal{B}(M)\right)\subset\mathbb{R}^E.$$

When M of rank r has a realization $V^*\hookrightarrow \Bbbk^E$, let us consider V^* as a point on the Grassmannian $Gr(r,E)$ of r-dimensional planes in $\Bbbk^E$. The standard action of the algebraic torus $T := (\Bbbk^*)^E$ on $\Bbbk^E$ induces an action of T on $Gr(r,E)$. One can thus consider the torus orbit closure of the point V^*, which is well-known to be isomorphic to the toric variety of the base polytope $Q(M)$ [Gel+87].

Matroids can be characterized in terms of base polytopes in the following way.

Proposition A.1.9. Let $\mathscr{S}$ be a collection of r-subsets of a finite set E, and define a polytope

$$Q_{\mathscr{S}} := \mathrm{Conv}\left(\sum_{i\in S}\mathbf{e}_i \colon S\in\mathscr{S}\right)\subset\mathbb{R}^E.$$

Then $Q_{\mathscr{S}}$ is a base polytope of a matroid if and only if every edge of $Q_{\mathscr{S}}$ is a parallel translate of $\mathbf{e}_i - \mathbf{e}_j$ for some $i \neq j \in E$.

We point to [GS87; Spe09; FS12] as some examples of studying the geometry of matroids through their base polytopes.

A.2 Linear operations and matroid operations

We describe three central matroid operations here, which are (i) direct sums, (ii) restrictions, and (iii) contractions.

Definition A.2.1. Let M and M' be matroids on E and E' (respectively). Then the **direct sum** $M \oplus M'$ is a matroid on $E \sqcup E'$ whose bases are $\{B \sqcup B' \colon B \in \mathcal{B}(M),\ B' \in \mathcal{B}(M')\}$.

If $\Bbbk^E \twoheadrightarrow V$ and $\Bbbk^{E'} \twoheadrightarrow V'$ are realizations of M and M' (respectively), then $\Bbbk^E \oplus \Bbbk^{E'} \twoheadrightarrow V \oplus V'$ is a realization of $M \oplus M'$. A matroid M is said to be **connected** if it is not a nontrivial direct sum of two or more matroids.

Definition A.2.2. Let M be a matroid on E, and A a subset of E. Then the **restriction** of M to A, denoted $M|_A$, is a matroid on A whose rank function determined by

$$\mathrm{rk}_{M|_A}(S) = \mathrm{rk}_M(S) \quad \text{for } S \subseteq A.$$

The **contraction** of M by A, denoted M/A, is a matroid on $E \setminus A$ whose rank function is determined by

$$\mathrm{rk}_{M/A}(S) = \mathrm{rk}_M(S \cup A) - \mathrm{rk}_M(A) \quad \text{for } S \subseteq E \setminus A.$$

If $v : \Bbbk^E \twoheadrightarrow V$ is a realization of M, then the restriction $v|_{\Bbbk^A} : \Bbbk^A \to v(\Bbbk^A)$ of the map v is a realization of the matroid $M|_A$. The map $\Bbbk^{E\setminus A} \simeq \Bbbk^E/\Bbbk^A \to V/V_A$ where $V_A = v(\Bbbk^A)$ is a realization of the matroid M/A.

Restriction and contraction by a flat of a matroid behave well with respect to the lattice of flats. If F is a flat of a matroid M, then the lattice of flats of the restriction $M|_F$ is isomorphic to the interval $[\hat{0}, F] \subset \mathscr{L}_M$ where $\hat{0}$ is the bottom element of $\mathscr{L}_M$. The contraction M/F has lattice of flats isomorphic to the interval $[F, E] \subset \mathscr{L}_M$.

If M has a realization $\Bbbk^{n+1} \twoheadrightarrow V$, so that we have a hyperplane arrangement $\mathcal{A}_M$ on $\mathbb{P}V^* \subset \mathbb{P}^n$, then the restriction and contraction by flats can be described as follows. For F a flat of M, let L_F be the corresponding linear subspace $\{f \in V^* \colon f(v_i) = 0\ \forall i \in F\}$. Then the intersections of $\mathbb{P}L_F$ with the hyperplanes of $\mathcal{A}_M$ not containing $\mathbb{P}L_F$ defines a hyperplane arrangement on $\mathbb{P}L_F$. This hyperplane arrangement on $\mathbb{P}L_F$ is a realization of the contraction M/F. Dually, consider the projection $\mathbb{P}V^* \dashrightarrow \mathbb{P}(V^*/L_F)$. The hyperplane arrangement on $\mathbb{P}(V^*/L_F)$, where the hyperplanes are the projections of the hyperplanes of $\mathcal{A}_M$ containing $\mathbb{P}L_F$, is a realization of the matroid $M|_F$.