ERRATA IN "THE GEOMETRY OF DIVISORS ON MATROIDS"

CHRISTOPHER EUR

Proposition 7.2.9 and Corollary 7.2.10 had an error that should be corrected as follows. No other part of the document is affected. Thanks to Matt Larson for the conversation that led to this correction.

Let us define the **signed beta invariant** of a matroid M of rank r to be

$$\widetilde{\beta}(M) = (-1)^{r-1}\beta(M),$$

which is also equal to $\overline{\chi}_M(1)$, the sum of coefficients of the reduced characteristic polynomial of M, by Theorem 7.2.8.(1). We can express the rank divisor class by the signed beta invariants in the following way.

Proposition 7.2.9. Let M be a loopless matroid of rank r on E. Then one has

$$\zeta_{Q(M)} = \sum_{\emptyset \subsetneq S \subseteq E} (-1)^{|S|}\widetilde{\beta}(M|_S) h_S, \text{ and thus } \zeta_{Q(M)}(M) = \sum_{F \in \mathscr{L}_M^{\geq 2}} \Big(\sum_{\mathrm{cl}_M(S) = F} (-1)^{|S|}\widetilde{\beta}(M|_S) \Big) h_F.$$

We also have $\zeta_{Q(M)}(M) = \sum_{F \in \mathscr{L}_M^{\geq 2}} \Big(-\sum_{\emptyset \subseteq G \subseteq F} \mu(G,F)\,\mathrm{rk}_M(G) \Big) h_F$.

Proof. We first recall the definition $h_F := \sum_{G \supseteq F} -z_F$. One computes that

$$(\dagger)\quad \text{if} \quad \sum_{F \in \mathscr{L}_M \setminus \{\emptyset\}} a_F h_F = \sum_{F \in \mathscr{L}_M \setminus \{\emptyset\}} b_F z_F, \quad \text{then}$$

$$\sum_{\emptyset \subsetneq G \subseteq F} -a_G = b_F, \quad \text{equivalently by Möbius inversion,} \quad a_F = \sum_{\emptyset \subsetneq G \subseteq F} -\mu(G,F) b_G.$$

We now compute:

$$\begin{aligned} \zeta_{Q(M)} &= \sum_S \mathrm{rk}_M(S) z_S \\ &= \sum_S \Big(-\sum_{\emptyset \subseteq T \subseteq S} (-1)^{|S \setminus T|}\,\mathrm{rk}_M(T) \Big) h_S \\ &= \sum_S \Big((-1)^{|S| + \mathrm{rk}(M|_S) + 1} \beta(M|_S) \Big) h_S \\ &= \sum_S (-1)^{|S|}\widetilde{\beta}(M|_S) h_S, \text{ and,} \end{aligned}$$

$$\begin{aligned} \zeta_{Q(M)}(M) &= \sum_F \mathrm{rk}_M(F) z_F \\ &= \sum_F \Big(-\sum_{\emptyset \subseteq G \subseteq F} \mu(G,F)\,\mathrm{rk}_M(G) \Big) h_F. \end{aligned}$$

In both cases, the first equality is Equation (7.1); the second equality is our observation ($\dagger$), where $\emptyset$ is included in the summation because $\mathrm{rk}_M(\emptyset) = 0$; and the third follows from either the definition of β. The first formula for $\zeta_{Q(M)}(M)$ follows from the one for $\zeta_{Q(M)}$ since $h_S(M) = h_{\mathrm{cl}_M(S)}(M)$. $\square$

A formula for the rank volume immediately follows by applying Theorem 5.2.4 to the above proposition, and is further simplified by applying Theorem 7.2.8.(2).

Corollary 7.2.10. Let M be a loopless matroid of rank $r = d+1$ on E. Then

$$\mathrm{RVol}(M) = \sum_{(F_1,\ldots,F_d)} \prod_{i=1}^{d} \Big(\sum_{\mathrm{cl}_M(S)=F_i} (-1)^{|S|} \widetilde{\beta}(M|_S) \Big)$$

where the summation is over all ordered sequences of flats $(F_1, \ldots, F_d)$ that satisfies $\mathrm{DHR}(M)$ and $M|_{F_i}$ is a connected matroid for each $1 \leq i \leq d$.