Lecture 8

Defn For $f \in \mathbb{R}[w_1, \cdots, w_m]$ homogeneous, $\operatorname{supp}(f) = \{ a \in \mathbb{Z}^m_{\geq 0} \mid w^a = w_1^{a_1} \cdots w_m^{a_m}$ has nonzero coeff. in $f\}$

Let $d\Delta_m = \{ a \in \mathbb{Z}^m_{\geq 0} \mid a_1 + \cdots + a_m = d \}$.

A subset $S \subseteq d\Delta_m$ is <u>M-convex</u> if $\forall\, a, b \in S$ and $1 \leq i \leq m$ with $a_i > b_i$, $\exists\, 1 \leq j \leq m$ st $a_j < b_j$ and $a - e_i + e_j,\ b + e_i - e_j \in S$.

E.g. A subset $S \subseteq d\Delta_m \cap [0,1]^m$ is M-convex iff the corresp. subsets of $\{1, \cdots, m\}$ form a matroid.

Thm The following subsets of the space of deg d homog. polynom. in $\mathbb{R}[w_1, \cdots, w_m]$ coincide.

(I) closure of the space of <u>strictly Lorentzian polynomials</u>

$$\left\{ \begin{array}{l} f \text{ st } \operatorname{supp}(f) = d\Delta_m, \text{ all coeff.s are positive, and} \\ \forall\, i_1, \cdots, i_{d-2},\quad \partial_{i_1}\partial_{i_2}\cdots\partial_{i_{d-2}} f \text{ has the Lorentzian signature } (+,-,\cdots,-) \end{array} \right\}$$

(II) $$\left\{ \begin{array}{l} f \text{ has nonneg. coeff. w/ M-convex support, and} \\ \forall\, i_1, \cdots, i_{d-2},\quad \partial_{i_1}\partial_{i_2}\cdots\partial_{i_{d-2}} f \text{ has at most one positive eigenval.} \end{array} \right\}$$

(III) the space of <u>completely log-concave polynoms</u>

$$\left\{ \begin{array}{l} f \text{ has nonneg. coeff., and every nonnegative directional partial derivative,} \\ \text{i.e. } (a_{11}\partial_1 + \cdots + a_{1m}\partial_m) \cdots (a_{k1}\partial_1 + \cdots + a_{km}\partial_m) f \text{ is } \equiv 0 \text{ or log-conc. on } \mathbb{R}^m_{>0} \end{array} \right\}$$

(IV) $\{ f \text{ st } (A_f^{\bullet}, \int, \mathcal{K}) \text{ satisfies mixed } HR^{\leq 1} \}$

pf) [Brändén–Huh '21], [Anari–Liu–Oveis Gharan–Vinzant '18].

N.B. $(a_0, a_1, \cdots, a_n)$ is a nonneg. log-conc. seq. w/ no internal zeroes

$$\Longleftrightarrow \quad a_0 \frac{x^n}{n!} + a_1 \frac{x^{n-1} y}{(n-1)!\,1!} + \cdots + a_i \frac{x^i y^{n-i}}{i!\,(n-i)!} + \cdots + a_n \frac{y^n}{n!} \quad \text{is Lorentzian.}$$

$$\left(\text{cf. } \int (\alpha x + \beta y)^n = n! \sum_{i=0}^{n} \left(\int \alpha^i \beta^{n-i} \right) \frac{x^i y^{n-i}}{i!\,(n-i)!} \right)$$

Prop If $\eta_1, \dots, \eta_m \in \overline{K(X)_{\mathbb{R}}}$ nef divisor classes on a proj. var. (not necessarily smth, or over char=0 field), then the VP w/r/t them is Lorentzian.

Ques Is there a Lorentzian polynomial that does not arise this way?

Thm [BH'21, Theorem 2.10] If $f(w_1, \dots, w_m)$ Lorentzian polynom., then $f(Av)$ also for any nonneg. matrix $A_{m \times m'}$. ($v = (v_1, \dots, v_{m'})$ new variables).

Cor Product of Lorentzian is Lorentzian

Conj. (Mason) Let M be a matroid on n elts, and write I_k for # indep. subsets of size k.

(i) $I_k^2 \geq I_{k-1} I_{k+1} \iff \sum_{i=0}^{n} I_i \frac{x^i y^{n-i}}{i!\,(n-i)!}$ is Lorentzian

(ii) $(k!\, I_k)^2 \geq (k-1)!\, I_{k-1} \cdot (k+1)!\, I_{k+1} \iff \sum_{i=0}^{n} I_i \frac{x^i y^{n-i}}{(n-i)!}$ Lorentzian

(iii) $\left(\frac{I_k}{\binom{n}{k}}\right)^2 \geq \frac{I_{k-1}}{\binom{n}{k-1}} \frac{I_{k+1}}{\binom{n}{k+1}} \iff I_0 y^n + I_1 x y^{n-1} + \cdots + I_n x^n$ Lorentzian

Thm [BH'21, ALOV'18] Mason's (iii) holds. Let M be a matroid on $[n] = \{1, 2, \dots, n\}$.

pf) Let $Z_M^q(w_0, w_1, \dots, w_n) = \sum_{S \subseteq [n]} q^{-rk_M(S)} w_S w_0^{n-|S|}$.

Claim: Z_M^q is Lorentzian for $0 < q \leq 1$.

(Note: Claim $\Rightarrow \lim_{q \to 0} Z_M^q(w_0, q w_1, \dots, q w_n) = \sum_{I \in \mathcal{I}_M} w_I w_0^{n-|I|}$).

pf) M-convex easy to check. $\frac{\partial}{\partial w_i} Z_M^q = q^{-rk_M(i)} Z_{M/i}^q$. By induction, need only check $\left(\frac{\partial}{\partial w_0}\right)^{n-2} Z_M^q = \frac{n!}{2} w_0^2 + (n-1)!\, w_0 \left(\sum_i q^{-rk_M(i)} w_i\right) + (n-2)! \left(\sum_{i<j} q^{-rk_M(ij)} w_i w_j\right)$.

I.e. need: $\left(\sum_i q^{-rk(i)} w_i\right)^2 - \frac{2n}{n-1} \left(\sum_{i<j} q^{-rk(ij)} w_i w_j\right) \geq 0 \quad \forall w_1, \dots, w_n \in \mathbb{R}$.

change of var. $\left(\sum_i w_i\right)^2 - \frac{2n}{n-1} \left(\sum_{i<j} q^{\rho(ij)} w_i w_j\right) \geq 0$ by $w_i \mapsto \begin{cases} w_i & \text{if loop} \\ q w_i & \text{if not.} \end{cases}$

$\rho(ij) = \begin{cases} 1 & i \parallel j \\ 0 & \text{else} \end{cases}$

Reduces eventually to $\frac{1}{n}\left(\sum_i w_i\right)^2 \leq \left(\sum_{i \in S_1} w_i\right)^2 + \cdots + \left(\sum_{i \in S_m} w_i\right)^2$ which follows from Cauchy–Schwartz.

$S_1 \sqcup \cdots \sqcup S_m = [n]$

Ques. If $L \subseteq \mathbb{C}^E$ realizes M, is there a proj var X_L & nef divisors $D_0, \dots, D_n$ st $VP = \sum_{I \in \mathcal{I}_M} w_I w_0^{n-|I|}$? (**Rem** There is such for Mason (ii)).