Lecture 5

Last time: deletion-contraction, $\chi_M(q) \approx$ Poincare polynom. of $\mathring{L} = L \cap (\mathbb{C}^*)^E$.

<u>Defn</u> For P a finite poset, the <u>Möbius fct</u> $\mu_P: P \times P \longrightarrow \mathbb{Z}$ is defined by

(1) $\displaystyle\sum_{x \le z \le y} \mu(x,z) = \begin{cases} 1 & \text{if } x=y \\ 0 & \text{if } x \lneq y \end{cases}$

(2) $\mu(x,y) = 0$ if $x \nleq y$

<u>Rem</u> (1') $\displaystyle\sum_{x \le z \le y} \mu(z,y) = \begin{cases} 1 \text{ if } x=y \\ 0 \text{ if } x \lneq y \end{cases}$ equivalent to (1). (A priori not obvious!)

$\mu(M) := \mu(\phi, E)$ is called the <u>Möbius invariant</u> of M. (top Whitney #'s of 1st kind)

<u>Prop</u> For $M = (E, \mathcal{F})$, $F \in \mathcal{F}$, and $S \subseteq F$, define $\mu(S,F) = \begin{cases} 0 & \text{if } S \notin \mathcal{F} \\ \mu(S,F) & \text{else.} \end{cases}$

Then, $\displaystyle\mu(S,F) = \sum_{\substack{S \subseteq X \subseteq F \\ cl_M(X) = F}} (-1)^{|X \setminus S|}$.

pf) If $S \notin \mathcal{F}$, $\exists e \in E \setminus S$ st $\overline{S} = \overline{S \cup e}$. Then for $S \subseteq X$, have $\overline{X \cup e} = \overline{X}$.

If $S \in \mathcal{F}$, $\displaystyle\phi(S,F) := \sum_{\substack{S \subseteq X \subseteq F \\ \overline{X} = F}} (-1)^{|X \setminus S|}$ satisfies $\displaystyle\sum_{S \subseteq G \subseteq F} \phi(S,G) = \sum_{S \subseteq X \subseteq F} (-1)^{|X \setminus S|} = \begin{cases} 1 & \text{if } S=F \\ 0 & \text{else.} \end{cases}$

<u>Cor</u> $\displaystyle\sum_{F \in \mathcal{F}} \mu(\phi, F)\, q^{r - rk_M(F)} = \sum_{X \subseteq E} (-1)^{|X|} q^{r - rk_M(X)} = \chi_M(q)$

<u>N.B.</u> (1) $(q-1)$ divides $\chi_M(q)$. $\rightsquigarrow$ $\overline{\chi}_M(q) := \chi_M(q)/(q-1)$ <u>reduced char pol.</u>

(2) $\chi_M = \chi_{simp(M)}$ if M loopless.

(3) $\deg \chi_M = r$ if M loopless.

(4) $\chi_M(q)$ has alternating sign coeff. ($\because$ Rota)

↳ if M is $\mathbb{C}$-realizable, follows from that $(-q)^r \chi(-\frac{1}{q}) =$ a Poincare polynom.

<u>Conj.</u> (Heron-Rota-Welsh '70s) |Coeff's| of χ_M form a log-concave sequence (in particular unimodal).

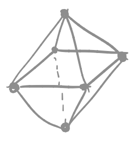
Thm To any $\mathbb{C}$-variety X, one can assign the <u>virtual Poincare polynom.</u> ${}^{v}P_X(q)$ determined by: (1) If X smth & cpt, ${}^{v}P_X = P_X$

(2) ${}^{v}P_X = {}^{v}P_U + {}^{v}P_{X\setminus U}$ $\quad U \subseteq X$ open.

E.g. ${}^{v}P_{(\mathbb{C}^*)^n} = (q^2-1)^n$, $\quad {}^{v}P_{\mathbb{C}^n} = (q^2)^n$

Rem Q a simplicial polytope $\Rightarrow$ toric variety X is a union of $f_i(\partial Q)$ many $(\mathbb{C}^*)^{\dim \partial Q - i}$ for $-1 \le i \le \dim \partial Q$.

$${}^{v}P_X(q) = \sum_{i \ge -1} f_i(\partial Q)(q^2-1)^{\dim\partial Q - i} = h_{\partial Q}(q^2)$$

 $\rightsquigarrow X = (\mathbb{P}^1)^3$

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      1
     1 6
    1 5 12
   1 4 7 8
1 3 3 1  ✓=  1 3 3 1
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Prop If $L \subseteq \mathbb{C}^E$ realizes M, then $\chi_M(q^2) = {}^{v}P_{\mathring{L}}(q)$

Thm (Möbius inversion) For fcts $f: P \to A$, $g: P \to A$, where A an abel. grp,

$$f(y) = \sum_{x \le y} g(x) \iff g(y) = \sum_{x \le y} \mu(x,y) f(x),$$

and

$$f(x) = \sum_{x \le y} g(y) \iff g(x) = \sum_{x \le y} \mu(x,y) f(y)$$

pf of Prop) For a flat F, let $\mathring{L}_F := L_F \setminus \left(\bigcup_{i \notin F} L_i\right)$. Then,

$${}^{v}P_L = {}^{v}P_\emptyset = \sum_{\emptyset \le F} {}^{v}P_{\mathring{L}_F} \iff {}^{v}P_{\mathring{L}_\emptyset} = \sum_{\emptyset \le F} \mu(\emptyset, F) \underbrace{{}^{v}P_{L_F}}_{=(q^2)^{r - rk_M(F)}}$$

Ques Other families of varieties st ${}^{v}P_X \approx P_X$?

Exer $\bar{\chi}_M(1) = \chi_{top}(\mathbb{P}\mathring{L})$ using Möbius inversion (no Gysin seq.)

Rem The <u>beta invariant</u> of a matroid $\beta(M) = (-1)^{r-1}\bar{\chi}_M(1)$ satisfies $\beta(M) \ge 0$ with $\beta(M) = 0$ iff M a loop or disconn.

<u>Defn</u> The <u>Tutte polynomial</u> of a matroid M of rank r on E is the bivariate polynom.

$$T_M(x,y) := \sum_{\phi \subseteq X \subseteq E} (x-1)^{r - rk_M(X)} (y-1)^{|X| - rk_M(X)}.$$

<u>N.B.</u> $T_{M^\perp}(x,y) = T_M(y,x)$ and $T_{M_1 \oplus M_2} = T_{M_1} \cdot T_{M_2}$.

<u>Thm</u> $T_M(x,y)$ satisfies the following (defining) property:

For $e \in E$, $T_M(x,y) = \begin{cases} y T_{M \setminus e} & \text{if } e \text{ a loop} \\ x T_{M/e} & \text{if } e \text{ a coloop} \\ T_{M \setminus e} + T_{M/e} & \text{if neither,} \end{cases}$ (*)

and $T_{U_{0,1}} = y$, $T_{U_{1,1}} = x$. $(T_{U_{0,0}} = 1)$

<u>Rem</u> Could've removed (*) and instead impose $T_{M_1 \oplus M_2} = T_{M_1} \cdot T_{M_2}$

<u>Cor</u> Suppose f is an invariant of matroids with values in a comm. ring R satisfying

(1) $\exists\, a, b \in R$ st $f(M) = a f(M \setminus e) + b f(M/e)$ if e neither loop/coloop, and

(2) $f(M) = \begin{cases} f(U_{0,1}) f(M \setminus e) & \text{if } e \text{ a loop} \\ f(U_{1,1}) f(M \setminus e) & \text{if } e \text{ a coloop.} \end{cases}$

(such f are called <u>Tutte-Grothendieck</u> invariants)

Then, $f(M) = a^{|E| - rk(M)} b^{rk(M)} T_M\left(\frac{x_0}{b}, \frac{y_0}{a}\right)$ where $x_0 = f(U_{1,1})$, $y_0 = f(U_{0,1})$

<u>E.g.</u> $(-1)^r T_M(1-q, 0) = \chi_M(q)$

$T_M(1,1) = \#$ bases, $T_M(2,1) = \#$ indep. subsets

[Brylawski-Oxley '92], [Welsh '99] for more.

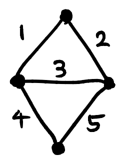
<u>Defn</u> Let $<$ be a total order on E. For a basis B of M,

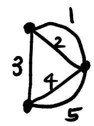
(1) $b \in B$ is <u>internally active</u> if e the min'l elt. of $E \setminus (\overline{B \setminus b})$

(2) $e \in E \setminus B$ is <u>externally active</u> if e the min'l elt. of the unique circuit in $B \cup e$.

Thm $$T_M(x,y) = \sum_{B \in \mathcal{B}} x^{|\mathrm{IntAct}(B)|}\, y^{|\mathrm{ExtAct}(B)|}$$

E.g.





M	M^*
123	12
1245	45
345	134
	135
	234
	235

$\mathcal{B}_M$	$x^I y^E$
124	x^3
125	x^2
134	x^2
135	x
145	xy
234	xy
235	y
245	y^2

$T_M = x^3+x^2+x+y + (x+y)^2$

$= x^3 + 2x^2 + 2xy + y^2 + x + y$

Ques (1) Geometric interpretation of T_M that makes positive coeff. apparent?

cf. χ_M has alt. sign coeff. for realizable matroid b/c betti # ≥ 0,

$\beta(M) := (-1)^{r-1}\overline{\chi}_M(1) \geq 0$ since $\Omega_{W_L}(\log \partial W_L)$ "almost" globally gen.

Log-concavity statements

(2) Explain / expand [Kochol'21] via internal-external activities