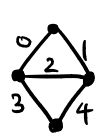
Lecture 3

Simplicial cplx $IN(M)$ specified by:

facets	$\longleftrightarrow$	bases
faces	$\longleftrightarrow$	indep. subsets
min'l non-faces	$\longleftrightarrow$	circuits

Defn A circuit of M is a minimal dependent (i.e. not indep.) subset of E.

E.g. If $M = M(G)$, circuit $\longleftrightarrow$ cycle whose vertices are traveled once



$\mathcal{C} = \{012, 234, 0134\}$

If $M = M\left(\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}\right)$ $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ -1 \end{bmatrix}$ (minimal supports of vectors in $\ker A$)

Thm A collection $\mathcal{C} \subset 2^E$ is the set of circuits of a matroid iff
(1) $\emptyset \notin \mathcal{C}$,
(2) $C_1 \subseteq C_2 \Rightarrow C_1 = C_2$ for $C_1, C_2 \in \mathcal{C}$, and
(circuit elimination) (3) for $C_1 \neq C_2 \in \mathcal{C}$ with $e \in C_1 \cap C_2$, $\exists C \in \mathcal{C}$ st $C \subseteq (C_1 \cup C_2) - e$.

Rem How would you "quickly" compute $\mathcal{C}$ from $\mathcal{B}$, and vice versa?
M2 currently uses that $I_{IN(M)}$, the Stanley-Reisner ideal of $IN(M)$,
is $\bigcap_{B \in \mathcal{B}} \langle x_i \mid i \notin B \rangle = \langle \prod_{i \in C} x_i \mid C \in \mathcal{C} \rangle$

Rem [Varbaro '11, Minh-Trung '11] $I_\Delta^{(m)}$ CM $\forall m \geq 1$ $\Longleftrightarrow$ $\Delta = IN(M)$ for some matroid M
pure & shellable $\Rightarrow$ $I_\Delta^\vee$ has linear resoln

Defn A subset is spanning in a matroid if it contains a basis.
A maximal non-spanning subset is a hyperplane of the matroid.

Exer If C a circuit and H a hyperplane, then $|C \setminus H| \neq 1$.

Defn The rank function $rk_M : 2^E \longrightarrow \mathbb{Z}_{\geq 0}$ of a matroid $M = (E, \mathcal{B})$ is defined by
$$rk_M(S) := \max\{|S \cap B| \mid B \in \mathcal{B}\}$$

E.g. If M realized as a list of vectors $(v_0, \dots, v_n)$, then $rk_M(S) = \dim \operatorname{span}(S)$.

Thm A function $f: 2^E \longrightarrow \mathbb{Z}_{\geq 0}$ is the rank function of a matroid iff:

(1) $f(S) \leq |S| \quad \forall S \subseteq E$,

(2) $f(S_1) \leq f(S_2)$ if $S_1 \subseteq S_2$, and

(3) $f(S_1) + f(S_2) \geq f(S_1 \cup S_2) + f(S_1 \cup S_2) \quad \forall S_1, S_2 \subseteq E$.

(submodularity)

$\Updownarrow$

diminishing marginal return [Schrijver '03]

N.B. $rk_M(S) = |S| \Leftrightarrow$ indep. , $\quad rk_M(S) = rk_M(E) \Leftrightarrow$ spanning

Defn A subset $F \subseteq E$ is a <u>flat</u> of M if it is max'l among subsets of rank $rk_M(F)$.
(i.e. $rk_M(F \cup e) > rk_M(F) \quad \forall e \in E \setminus F$)

E.g. ϕ is a flat of M iff M loopless.

E.g. If G conn. graph, then flats of corank c in $M(G) \longleftrightarrow$ Partitions of G into $c+1$ components.
($V = \bigsqcup_i V_i$ st $G|V_i$ conn.)

E.g. Let $k^E \twoheadrightarrow L^\vee$ realize a loopless matroid M, where v_i = image of e_i $(i \in E)$.

a flat of rank m $\longleftrightarrow$ $\{i \in E \mid v_i \in V\}$ for some m-dim'l subspace $V \subseteq L^\vee$.

i.e. $\{\text{flats of rk } m\} \longleftrightarrow \{m\text{-dim'l spans of subsets of } v_0, \dots, v_n \text{ in } L^\vee\}$

$$\longleftrightarrow \{m\text{-}\underline{\text{codim'l}} \text{ subspaces in } L\}$$

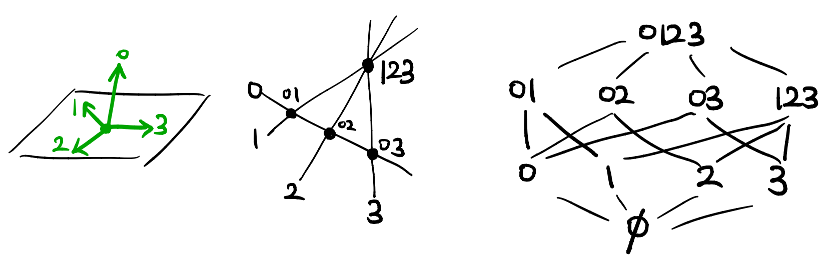
$$F \longmapsto \{\ell \in L \mid v_i(\ell) = 0 \ \forall i \in F\}$$

Notation: $L_F := \{\ell \in L \mid v_i(\ell) = 0 \ \forall i \in F\} = \bigcap_{i \in F} L_i \qquad (L_\phi = L)$.

$$\mathring{L} = L \setminus \bigcup_{i \in E} L_i = L \cap (k^*)^E$$

$$\simeq \mathbb{C}^* \times \mathbb{P}\mathring{L}$$

E.g.



Thm A collection $\mathcal{F} \subseteq 2^E$ is the subset of flats of a matroid iff:

(1) $E \in \mathcal{F}$

(2) $F_1, F_2 \in \mathcal{F} \Rightarrow F_1 \cap F_2 \in \mathcal{F}$

(3) For any $F \in \mathcal{F}$ and $e \in E \setminus F$, $\exists!\ F'$ covering F & containing e.

(3′) For any $F \in \mathcal{F}$, $\{F' \setminus F \mid F' \text{ covers } F \text{ in } \mathcal{F}\}$ partitions $E \setminus F$.

(Here, F' covers F if $F \subseteq F'$ and $\nexists G \in \mathcal{F}$ with $F \subsetneq G \subsetneq F'$)

Rem The poset of flats of a matroid (by inclusion) is

(1) graded (i.e. every max'l chain has length r)

(2) a lattice (i.e. $x \wedge y$ (GLB) and $x \vee y$ (LUB) exist)

that is geometric $\begin{cases} \text{semimodular } (x, y \text{ cover } x \wedge y \Rightarrow x \vee y \text{ cover } x, y) \\ \& \\ \text{atomic (every elt is a join of atoms)} \end{cases}$

Exer (1) If $e \notin B$ for a basis B, $B \cup e$ contains a unique circuit (called the fundamental circuit of (B, e)).

(2) If $S \subseteq S'$, then $\overline{S} \subseteq \overline{S'}$ and $\overline{S \cup S'} = \overline{\overline{S} \cup \overline{S'}}$.

(3) Use previous parts to show the strong exchange axiom:

If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \setminus B_2$, then $\exists y \in B_2 \setminus B_1$ such that both $B_1 - x \cup y$ and $B_2 - y \cup x$ are in $\mathcal{B}$.

Defn The <u>dual matroid</u> $M^\perp$ of a matroid $M = (E, \mathcal{B})$ is the matroid on E with bases $\{E \setminus B \mid B \in \mathcal{B}\}$. $((M^\perp)^\perp = M)$

Exer If the columns of $[A]$ represent M, then $[A^\perp]$ whose rows span the kernel of A represent $M^\perp$.

Ques. Let $\mathcal{M}_{r,n} :=$ # of rank r matroids on $[n] = \{1, 2, \cdots, n\}$.

$\mathcal{M}_{r,n}$ is symmetric by duality.

Open problem: Is $\mathcal{M}_{r,n}$ unimodal? (Probably not log-conc.)

Is there a Poincare duality explaining symmetric-ness?

TABLE 15.2. The number of non-isomorphic rank-r matroids on an n-set.

(count w/o modding out isom.)

r \ n	0	1	2	3	4	5	6	7	8	9
0	1	1 1	1 1	1 1	1 1	1 1	1 1	1	1	1
1		1 1	2 3	3 7	4 15	5 31	6 63	7	8	9
2			1 1	3 7	7 36 •	13 171 •	23 813 •	37	58	87
3				1 1	4 15	13 171 •	38 2053 •	108	325	1275
4					1 1	5 31	23 813 •	108	940	190214
5						1 1	6 63	37	325	190214
6							1 1	7	58	1275
7					• 33	• 131	• 473	1	8	87
8						• 131	• 883		1	9
9							• 473			1

Rem We'll later compare green #s to h-vectors of stellahedra.

$H^\bullet(X_{St_n})^{\mathfrak{S}_n}$ has dim 2^n w/ betti seq. $\binom{n}{i}$, and is related to $H^\bullet$ of a Hessenberg var.