Concluding Remarks

M a matroid of rank r on E.

- Many cryptomorphisms (flats, circuits, indep., polytope, etc.)

- Realizability of M, i.e. $L \in Gr(r,E)$ st $M(L) = M$.
  - $U_{2,4}$ not realizable over $\mathbb{F}_2$. The only forbidden minor.

    Thm (Tutte) M realizable over $\mathbb{F}_2 \iff M$ has no minor $\simeq$ to $U_{2,4}$

    (Bixby, Seymour) M realizable over $\mathbb{F}_3 \iff$ ——//—— $U_{2,5}, U_{3,5}$, Fano, $\text{Fano}^\vee$.

    (Tutte) M realizable over every field $\iff$ ——//—— $U_{2,4}$, Fano, $\text{Fano}^\vee$.

    Conj. (Rota) Thm (Geelen-Gerards-Whittle) (announced ~'14) # forbidden minors for realizability over $\mathbb{F}$ is finite for all $|\mathbb{F}| < \infty$.

  - Algebraic matroids: L/K field ext. $E = \{\ell_1, \dots, \ell_n\} \subset L$, $\mathcal{I} =$ alg. indep. subsets.

    E.g. Non-Pappus is alg. over $\mathbb{F}_2$ (Lindström): $\{x, x+y, y, \frac{xz}{x+y}+x+y, z, \frac{yz}{x+y}+z+y, xz, \frac{xyz}{x+y}+xy, yz\}$

    Ques. Is the dual of an alg. matroid algebraic?

  - Mnëv's universality (L. Lafforgue version:)

    Thm Let $X = \mathbb{Z}[\underline{x}]/I=\langle f_1, \dots, f_m\rangle$ and $p \in X_k$. Then $\exists$ a matroid strata $Y \subset Gr(r,E)(k)$, $Y \ni q$ ($r=3$ enuf) st (Y,q) is stably eqv. to (X_k, p).

    Mnëv: (Analogous statement holds for semi-alg. sets & oriented matroids).

- Log-concavity: mixed $HR^{\leq 1}$: Lorentzian polynom. = M-convex (polymatroid) + $\partial^{d-2}f$ signatures

  $\cap$

  Kähler package: Σ_M is Lefschetz, Lefschetz-ness dep. only on supp.

  Combinatorial need: invar. of matroid as an intersection #.

  Ques. Recall that $\sum_{I \text{ indep.}} w_I w_0^{|E|-|I|}$ is Lorentzian.

  If M realizable, is this a volume polynomial on a variety?

  Conj. (Brylawski) Coeff.s of $\frac{T_M(x,0)}{x}$ satisfy $\frac{c_k^2}{\binom{n-k}{crk-1}^2} \geq \frac{c_{k+1}}{\binom{n-k-1}{crk-1}} \frac{c_{k-1}}{\binom{n-k+1}{crk-1}}$

  Rem $W_k := \#(\text{rk}=k \text{ flats of } M)$. Unimodal? (Up to halfway & top-heavy).

- Tautological bundles/classes of matroids: $\mathcal{S}_L$, $\mathcal{Q}_L$
  - Chern classes ⇝ Tutte polynom.

  <u>Ques.</u> Schur classes ?

  <u>Ques.</u> $H^i(\Lambda^i \mathcal{S}_L^{(\vee)} \otimes \Lambda^j \mathcal{Q}_L^{(\vee)}) = ?$ Does it only depend on $M(L)$?