Lecture 22

Classical Hirzebruch-Riemann-Roch thm:

$$\begin{tikzcd} K(X)_{\mathbb{Q}} \arrow[r, "ch"] \arrow[d, "\chi"'] \arrow[dr, phantom, "\circlearrowright"] & A^\bullet(X)_{\mathbb{Q}} \arrow[d, "\int_X (-)\cdot Td(X)"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd}$$

$ch([\mathcal{O}_X(D)]) = \exp(D) = 1 + D + \frac{D^2}{2!} + \frac{D^3}{3!} + \cdots$

via $\frac{x}{1-e^{-x}}$

(hard to work with!)

<u>Thm</u>

$$\begin{tikzcd} \zeta : K(X_E) \arrow[r, "\sim"] \arrow[d, "\chi"'] & A^\bullet(X_E) \arrow[d, "\int_{X_E}(-)\cdot(1+\alpha+\cdots+\alpha^n)"] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd}$$

pf) KEY: $K_T \longrightarrow A^\bullet_T[\prod_i (1+t_i)^{-1}]$ $\qquad A^\bullet_T(pt) = \mathrm{Sym}^\bullet \mathrm{Char}(T)$

$T_i \longmapsto 1+t_i$

$\equiv 0 \mod (1-\frac{T_i}{T_j}) \iff \equiv 0 \mod (\underbrace{T_i - T_j}_{t_i - t_j})$

<u>Prop</u> ① $\zeta[\underbrace{\mathcal{O}_{X_E}(D_{-P(M)})}_{\cong \det S_M^\vee}] = c(S_M, -1) = \sum_{i\geq 0} c_i(S_M)(-1)^i$

$$\sum_{i\geq 0} \zeta(\Lambda^i Q_M^\vee)u^i = (u+1)^{|E|-r} c(Q_M^\vee, \tfrac{u}{u+1})$$

② $\zeta[\mathcal{O}(\alpha)] = \frac{1}{1-\alpha}$, $\quad \zeta[\mathcal{O}(\beta)] = 1+\beta$, $\quad \zeta[\mathcal{O}(\underbrace{h_S}_{= D_{\mathrm{Conv}(e_i \mid i\in S)}})] = \frac{1}{1-h_S}$

<u>N.B.</u> $\mathcal{O}_{W_L}$ the structure sheaf of $W_L = V(s_{\mathbb{1}}) \hookrightarrow X_E$ for $s_{\mathbb{1}} \in H^0(Q_L)$.

$$\cdots \longrightarrow \Lambda^2 Q_L^\vee \longrightarrow Q_L^\vee \longrightarrow \mathcal{O}_{X_E} \longrightarrow \mathcal{O}_{W_L} \longrightarrow 0$$

$$\Rightarrow \zeta[\mathcal{O}_{W_L}] = c_{|E|-r}(Q_L) = [W_L]$$

Recall: Schubert matroids

<u>Cor</u> $\overline{\mathbb{I}}(\mathrm{SchMat}_n) \hookrightarrow \overline{\mathbb{I}}(\Sigma_E) \longrightarrow K(X_E)$ seq. of isom.

($\Rightarrow$ loopless Sch. mat. form a basis).

<u>Cor</u> [Thm 11.3, Postnikov '09]

Let $[D_{P_y}] = (y_E - 1)h_E + \sum_{S \subsetneq E} y_S h_S$.

Then $\Upsilon(\mathrm{Vol}(P_y)) = \#|P_y \cap \mathbb{Z}^n|$

$\Upsilon : \frac{x^d}{d!} \longmapsto \binom{x+d-1}{d}$

<u>Cor</u> [Cameron-Fink '18] and [Fink-Speyer '12].

<u>Defn</u> For $\mathcal{E}$ vec. bdl on X, let $\mathbb{P}_X(\mathcal{E}) := \mathcal{P}roj\ \mathrm{Sym}^\bullet \mathcal{E}^\vee$, and let $h = c_1(\mathcal{O}(1))$ for the $\mathcal{O}(1)$ on $\mathbb{P}_X(\mathcal{E})$. Let $\pi : \mathbb{P}_X(\mathcal{E}) \longrightarrow X$.

<u>E.g.</u> For
$$\begin{tikzcd} S \arrow[rr, hook] \arrow[dr] & & \underline{\mathbb{C}}^N \arrow[dl] \\ & X & \end{tikzcd}$$
, the $\mathcal{O}(1)$ on $\mathbb{P}_X(S)$ gives the map $\mathbb{P}_X(S) \longrightarrow \mathbb{P}_{\mathbb{C}}^{N-1}$.

$(x, \overline{v}) \longmapsto \overline{v}$.

<u>Defn</u> The ith-<u>Segre class</u> of $\mathcal{E}$ is $s_i(\mathcal{E}) = \pi_*(h^{\mathrm{rk}(\mathcal{E})-1+i}) \in A^i(X)$.

<u>Prop</u> The total Segre class $s(\mathcal{E}) = 1 + s_1(\mathcal{E}) + s_2(\mathcal{E}) + \cdots$ satisfy

$$s(\mathcal{E})\, c(\mathcal{E}) = 1 \quad \text{in} \quad A^\bullet(X).$$

<u>N.B.</u> $A^\bullet(\mathbb{P}_X(\mathcal{E})) \simeq A^\bullet(X)[h] \big/ \langle h^r + h^{r-1}c_1(\mathcal{E}) + \cdots + c_r(\mathcal{E}) \rangle$

$$1 = \pi_*(h^{r-1}) = \pi_*\left(\frac{1}{1-h} c(\mathcal{E})\right) = \pi_*\left(\frac{1}{1-h}\right) \cdot c(\mathcal{E}) = s(\mathcal{E}) c(\mathcal{E})$$