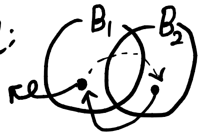
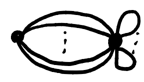
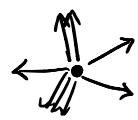
Lecture 2. (M matroid on E with bases $\mathcal{B}$ of rank r).

Recall:  $\longleftrightarrow$ "equations" for $Gr(r,E)(\mathbb{K})$

Realizable/Linear matroids $\longleftrightarrow$ vector config. in k-vec. sp. $\longleftrightarrow$ pts in $Gr(r,E)$

$\boxed{\text{Rank } 1}$ matroids: $M = M($  $)$ (realizable over any field).

$e \in E$ a <u>loop</u> if no basis of M contains e.

$\boxed{\text{Rank } 2}$ matroids: $M = M($  $)$ (realizable over any char, but not any field)

<u>Prop</u> rank 2 matroid M has $E = (\text{loops}) \sqcup E_1 \sqcup \cdots \sqcup E_m$ st $\mathcal{B} = \{ab \mid a \in E_i, b \in E_j, i \neq j\}$.

pf) If loopless, then ($i \sim j$ if ij not a basis) is an equiv. rel.

$\cancel{ij}$ jk

ik $\underset{\text{basis}}{jl}$ (can't exchange l)

E_i's <u>parallel classes</u>

<u>Defn</u> Let M (arbitrary rank) loopless matroid. Say i, j <u>parallel</u> ($i /\!/ j$) if ij not in any basis of M.

<u>Exer</u> (1) $i \sim j$ if $i /\!/ j$ is an equiv rel.

(2) $i \in B$ & $i /\!/ j$ then $B \cup j - i$ basis also.

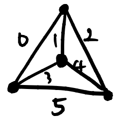
<u>Defn</u> M a matroid. <u>Simplification</u> of M: remove loops, then collapse each parallel class to a single elt.

M <u>simple</u> if equal to its simplification.

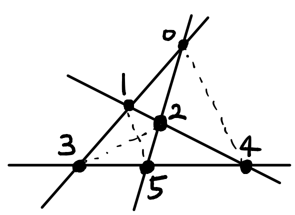
$\boxed{\text{Rank } 3}$ simple matroids: Points-lines diagrams

A <u>line</u> in a rank 3 matroid is a max'l not-basis-containing subset.

<u>E.g.</u> $G = K_4 \rightsquigarrow M(G)$ on $E = \{0, 1, \cdots, 5\}$ has 7 lines: 013, 124, 345, 025, 04, 15, 23



If were to realize $M(K_4)$ as pts in $\mathbb{P}^2$, have

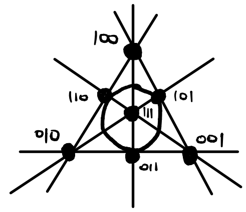


<u>Defn</u> Points-lines diagram of a (simple) rank 3 matroid M: pts for each E, lines for each line w/ ≥ 3 pts.

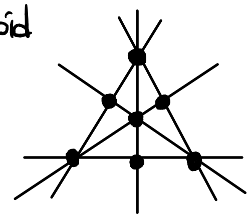
⚠ not to confuse w/ illustrations of hyperplane arr.

E.g. (Pts in $\mathbb{P}^2_{\mathbb{F}_2}$) (Realizable only over char $=2$)

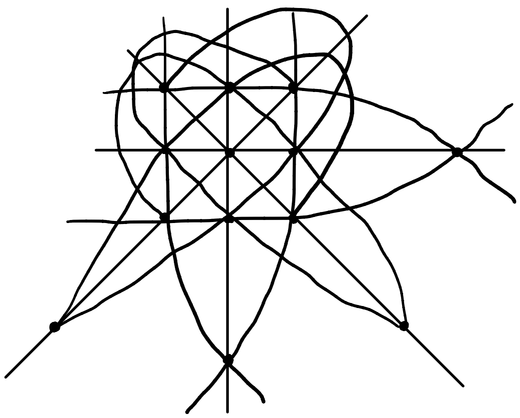
<u>Fano matroid</u>



E.g. <u>non-Fano matroid</u> (realizable only over char $\neq 2$)



E.g. $\mathbb{P}^2_{\mathbb{F}_3}$



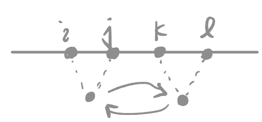
Prop Let $\mathcal{L}$ be a collection of subsets of E, called <u>lines</u>, such that:

(1) Every two elts of E in a line,
(2) Every line has at least two elts,
(3) Two lines intersect in at most one elt,
(4) $\exists$ 3 elts not in a line.

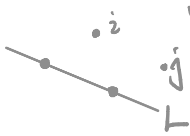
Then $\mathcal{B} = \left\{ B \in \binom{E}{3} \;\middle|\; B \text{ not in a line} \right\}$ defines a rank 3 matroid, and every rank 3 simple matroid on E arises in this way.

pf) Show $\mathcal{B} \rightleftarrows \mathcal{L}$ well-def. & are inverses

(i) For $\mathcal{B} \longrightarrow \mathcal{L}$: rank 3 $\Rightarrow$ (1), (2), (4). (3): (2 pts on a line) $\cup$ (a pt not on the line) is a basis, since

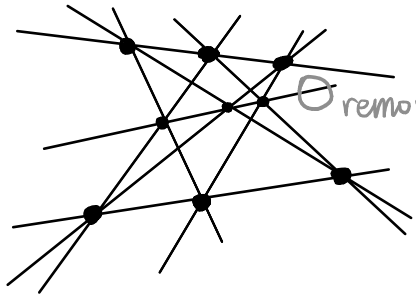


(ii) For $\mathcal{L} \longrightarrow \mathcal{B}$: If $i \in B_1 \setminus B_2$, take $j \in B_2 \setminus L$ where $L =$ the line containg $B_1 \setminus i$.



Inverse immediate to check.

E.g.



$\bigcirc$ remove $\rightsquigarrow$ non-Pappus matroid (not realizable)

Rem [Nelson '18] Almost all matroids are not realizable, i.e. $\dfrac{\#\text{realizable mat. on } n\text{-elts}}{\#\text{mat. on } n\text{-elts}} \longrightarrow 0$ as $n \to \infty$.

Every rank 2 matroid is realizable over any char.

Every rank 3 on 6 elts also.

—— // —— 8 elts is realizable over some field. [Fournier '71]

Vamos matroid: rank 4 on 8 elts, not realizable.

Defn $I \subseteq E$ is an independent subset in $M = (E, \mathcal{B})$ if $\exists B \in \mathcal{B}$ st $I \subseteq B$.

Thm A collection $\mathcal{I} \subseteq 2^E$ is the set of indep. subsets of a matroid on E iff:

(1) $\phi \in \mathcal{I}$

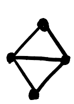
(2) downward closed: $I \subset J$ and $J \in \mathcal{I} \Rightarrow I \in \mathcal{I}$.

(3) If $I, J \in \mathcal{I}$ st $|I| < |J|$, then $I \cup j \in \mathcal{I}$ $\exists j \in J \setminus I$.

matroid indep. cplx $IN(M)$ has vertices $= E$, faces $=$ indep. subsets, pure of $\dim = r-1$.

Conj. (a) [Welsh '71, Mason '72] The f-vector $(f_{-1}, f_0, \cdots, f_{r-1})$ of $IN(M)$ is log-conc. (unimodal)

(b) [Dawson '84] The h-vector of $IN(M)$ is log-conc. (nonneg. by shellability)

E.g. $M = M\Big($  $\Big)$ $\quad (f_{-1}, \cdots, f_3) = (1, 5, 10, 8)$. $\quad f(q) = \sum_{i \geq -1} f_i \cdot q^{r-1-i}$

$h(q) = f(q-1)$.

1

1 5

1 4 10

1 3 6 8

1 2 3 2

Rem. Analogous statements fail for boundary cplxs of simplicial polytopes.

Exer Prove (a) for rank 3 matroids. (Hard already at rank ≥ 4.)