Lecture 19

[Ardila] 3 models: Base polytope, Bergman fan, Conormal fan $\Big\} \rightsquigarrow$ Tautological classes of matroids

<u>Today</u>: Tautological bundles of linear matroids

Let $E = \{0, 1, \cdots, n\}$, and $L = \text{rowspan}[\ A\ ] \subseteq \mathbb{C}^E$ realizing M of rk r.
Let X_E be the permutohedral var. of $\dim = n$ (i.e. X_Σ where $\Sigma = \Sigma_{A_n}$)
Let $T = (\mathbb{C}^*)^E$. X_E is a T-variety (a $\mathbb{P}T = (\mathbb{C}^*)^E/\mathbb{C}^*$ - toric variety)

<u>Defn</u> The tautological sub / quotient bundles of L are T-eqv. vect. bndls on X_E
$\mathcal{S}_L$ whose fiber over $t \in \mathbb{P}T \subset X_E$ is $t^{-1}L$.
$\mathcal{Q}_L$ ——— // ——— is $\mathbb{C}^E/t^{-1}L$.

<u>Exer</u> Verify that $\mathcal{S}_L$ & $\mathcal{Q}_L$ are well-defined by considering limits over elts in one-parameter subgrp.

<u>N.B.</u> For $u \in N = \text{Cochar}(\mathbb{P}T)$, $u \in \text{relint}(\sigma)$, have $\lambda^u : \mathbb{C}^* \hookrightarrow \mathbb{P}T$.
$\lim_{t \to 0} \lambda^u(t) =$ the "1" in $O(\sigma)$.

<u>N.B.</u> $0 \to \mathcal{S}_L \to \underline{\mathbb{C}}^E = X_E \times \mathbb{C}^E \to \mathcal{Q}_L \to 0$

Equivalently, let $T \curvearrowright \mathbb{C}^E$ by $t \cdot v = t^{-1} v \rightsquigarrow T \curvearrowright Gr(r; E)$

$$\begin{tikzcd} X_E \arrow[d] \arrow[drr, "\varphi_L"] & & 0 \to \mathcal{S} \to \underline{\mathbb{C}}^E \to \mathcal{Q} \to 0 \arrow[d, dash] \\ X_{-P(M)} \simeq \overline{T \cdot L} \arrow[rr, hook] & & Gr(r, E) \end{tikzcd}$$

$\mathcal{S}$: fiber over L is $L \subseteq \mathbb{C}^E$

$\det \mathcal{Q} = \mathcal{O}(1)$ of Plücker embedding

$\varphi_L^* \mathcal{S} = \mathcal{S}_L$, $\quad \varphi_L^* \mathcal{Q} = \mathcal{Q}_L$

<u>Defn</u> $\mathcal{F}$ glob. gen. vec. bdl on X smth variety of rank ℓ. For $0 \leq i \leq \ell$,
$$c_i(\mathcal{F}) := \big[\{x \in X \mid (s_1(x), \cdots, s_{\ell+1-i}(x)) \text{ linearly dep.}\}\big] \in A^i(X)$$
for $s_1, \cdots, s_{\ell+1-i}$ general global sections of $\mathcal{F}$.

$$c(\mathcal{F}) := 1 + c_1(\mathcal{F}) + \cdots + c_\ell(\mathcal{F}), \quad c(\mathcal{F}, w) = 1 + c_1(\mathcal{F}) w + \cdots + c_\ell(\mathcal{F}) w^\ell$$

E.g. $c_\ell(\mathcal{F}) = [V(s)]$

$c_1(\mathcal{F}) = [V(\det(s_1, \cdots, s_\ell))] = c_1(\Lambda^\ell \mathcal{F} = \det \mathcal{F})$

Thm (1) $f: X' \longrightarrow X$ then $f^* c(\mathcal{F}) = c(f^*\mathcal{F})$

(2) $0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$ then $c(\mathcal{F}) = c(\mathcal{F}')\, c(\mathcal{F}'')$

(3) [Poincaré – Hopf] $c(X) := c(\mathcal{T}_X)$ "characteristic class" of X.

If X smth proj. $\mathbb{C}$-var, $\int_X c_n(\mathcal{T}_X) = \chi_{top}(X)$.

① $c_1(Q_L) = -c_1(\mathcal{S}_L) = D_{-P(M)}$ $\quad (\because \det Q_L = \varphi_L^*(\det Q))$

② $c_{|E|-r}(Q_L) = [W_L = \overline{\mathbb{P}L^\circ} \text{ in } X_E]$

Take $s_1 : X_E \longrightarrow \underline{\mathbb{C}}^E \longrightarrow Q_L$, "general" section.

$x \longmapsto (x, \mathbb{1})$

$V(s_1) = \{ x \in X_E \mid \mathcal{S}_L|_x \ni \mathbb{1} \}$

$\Updownarrow$ for $x = \bar{t} \in \mathbb{P}T \subset X_E$

$t^{-1}L \ni \mathbb{1} \iff t \in L$ $\quad \Rightarrow V(s_1) \cap \mathbb{P}T = \mathbb{P}L^\circ$

③ (Koszul cplx $\Lambda^\bullet Q_L^\vee \longrightarrow \mathcal{I}_{W_L/X_E} \longrightarrow 0) \Rightarrow Q_L|W_L = N_{W_L/X_E}$

$0 \longrightarrow \mathcal{T}_{W_L} \longrightarrow \mathcal{T}_{X_E} \longrightarrow N_{W_L/X_E} \longrightarrow 0$

$\Rightarrow \quad K_{W_L} = (D_{-P(M)} + \underbrace{K_{X_E}}_{= -\alpha-\beta})|_{W_L} \Rightarrow$ log can. div. $= D_{-P(M)}$