Lecture 18

Rem. on proofs of the two fund. thms in trop. Hodge thry:

Thm A. $|\Sigma_1| = |\Sigma_2|$ then Σ_1 Lefschetz $\Leftrightarrow$ Σ_2 Lefschetz.

Thm B. Σ_M is Lefschetz for any loopless matroid M.

Cor [HLSW'22?] Hodge thry of polymatroids [Pagaria-Pezzoli '21]

Lem (1) $HL^i + (HR^i \text{ at } \ell \in \overline{K}) \Rightarrow HR^i$ for all K.

(2) If $A^\bullet = \mathbb{R}[x_1, \dots, x_m]/I$ P.D. alg. w/ $\int_A$ and $\ell \in \mathbb{R}_{>0}\{x_1, \dots, x_m\}$, then $(HL^i + HR^i$ for all $A^\bullet/\mathrm{ann}(x_j)$ at $\ell_j) \Rightarrow HL^i$ for $A^\bullet$ at ℓ.

pf) $\int \ell^{d-2i} f^2 = \sum_j \int_j c_j f_j^2 \ell_j^{d-2i-1} = 0$ iff $f_j = 0$ when $f\ell^{d-2i} = 0$ (since f_j prim.)

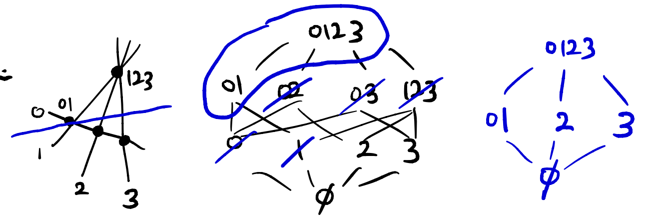
Induction I. ① $A^\bullet(\Sigma_M)/\mathrm{ann}(x_F) \simeq A^\bullet(\Sigma_{M|F}) \otimes A^\bullet(\Sigma_{M/F})$

② Let $\alpha := \sum_{F \ni i} x_F$ for any $i \in E$, and define $h_F = \alpha - \sum_{G \supseteq F} x_G$.

$A^\bullet(\Sigma_M)/\mathrm{ann}(h_F) = A^\bullet(\Sigma_{Tr_F(M)})$ where

$\mathcal{B}(Tr_F(M)) = \{B \setminus f : B \in \mathcal{B}(M),\ f \in B \cap F \neq \phi\}$

E.g.



012, 013, 023   02, 03, 12, 13, 23

Rem $\Delta_{Tr_F(M)} = \Delta_M \cap_{st} \Delta_{H_F}$ $\mathcal{B}(H_F) = \{B \in \binom{E}{|E|-1} \mid B \not\supseteq F\}$

Thm [Backman-E.-Simpson '19] [Postnikov '09] $h_{F_1} \cdots h_{F_{r-1}} \cap \Delta_M = \begin{cases} 1 & \text{if } \forall \phi \subsetneq J \subseteq [r-1],\ rk_M(\bigcup_{j \in J} F_j) \geq |J|+1 \\ 0 & \text{else} \end{cases}$

Does $F_1, \dots, F_{r-1}$ admit a transversal in $E \setminus i$ that is indep. in M for any $i \in E$?

Rem [Dastidar-Ross '21] Matroid Psi classes: h_F's are pullbacks of some psi classes from $X_{A_n} = LM_{n+1} = \{(C, (0, \infty, p_0, \dots, p_n))$
$(1, 1, \varepsilon, \dots, \varepsilon)$

Induction II.

Defn $f: X \twoheadrightarrow Y$ proper surj., X smth, is <u>semi-small</u> if
$$\mathrm{codim}_Y (S_k := \{y \in Y \mid \dim f^{-1}(y) = k\}) \geq 2k \quad \forall k.$$

Thm [de Cataldo – Migliorini '02] Let ℓ be ample & base-pt-free on Y, and $f: X \twoheadrightarrow Y$ proper surj. w/ X smth. Then $f^*\ell$ satisfies HL & HR iff f semi-small.

E.g. If $Z \subset Y$ smth subvar. of smth Y of $\mathrm{codim}_Y Z = c$, then
$$\pi: X = Bl_Z Y \twoheadrightarrow Y \text{ semi-small iff } c = 2.$$

E.g. ① $Bl_L \mathbb{P}^3 = X$
$$\downarrow$$
$$\mathbb{P}^3 = Y$$

$$A^\bullet(X) = \oplus = \frac{\mathbb{Z}[h,e]}{\langle h^2 e, (h-e)^2 \rangle}$$
$$\pi^* A^\bullet(Y) \oplus (A^\bullet(Z) \otimes e)$$

h^3	
h^2	he
h	e
1	

(semi-small)

② $Bl_{pt} \mathbb{P}^3 = X$
$$\downarrow$$
$$\mathbb{P}^3 = Y$$

$$A^\bullet(X) = \oplus \oplus = \frac{\mathbb{Z}[h,e]}{\langle he, h^3 - e^3 \rangle}$$

h^3	
h^2	e^2
h	e
1	

(not semi-small)

For Thm A, weak factorization via edge subdivisions: [Abramovich-Karu-Matsuki-Włodarczyk '02]
$\exists\ \Sigma_1 = \Sigma^{(0)}, \ldots, \Sigma^{(m)} = \Sigma_2$ s.t. $\Sigma^{(i)}$ & $\Sigma^{(i+1)}$ related by the stellar subdiv. of a 2-dim'l cone.

For Thm B, if $i \in E$ not a coloop, then $\Sigma_M \to \Sigma_{M \setminus i}$ analogue of semi-small map.
$$\mathbb{R}^E/\mathbb{R}\mathbb{1} \to \mathbb{R}^{E \setminus i}/\mathbb{R}\mathbb{1}$$
[Braden–Huh–Matherne – Proudfoot–Wang '20]

E.g. $\mathcal{B}(M) = \binom{\{0,\ldots,4\}}{4} \setminus \{1234\}$ (note $X_{A_n} \to X_{A_{n-1}}$ has 1-dim'l fibers).
(actually $LM_{n+1} \twoheadrightarrow LM_n$ is the univ. curve)

$\mathbb{P}^3$