Lecture 14

$\Sigma = \Sigma_P$ a smth proj. fan for P full dim'l. $X = X_\Sigma$.

$$\mathrm{Pic}(X) = A^1(X) = \mathbb{Z}\{D_\rho \mid \rho \in \Sigma(1)\} \Big/ \mathbb{Z}\{\textstyle\sum_\rho \langle m, u_\rho\rangle D_\rho \mid m \in M\}$$

$$\simeq \{\text{piecewise linear fcts } \varphi_D \text{ (w/ integral slopes)}\} \Big/ \{\text{linear fcts}\}$$

$$A^1(X)_{\mathbb{R}} \supset \mathrm{Nef}(X) = \{\varphi_D \text{ PL fct} \mid \varphi_D(u+u') \leq \varphi_D(u) + \varphi_D(u')\}.$$

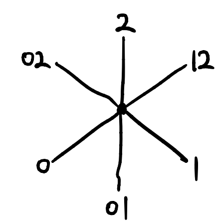
E.g. Let $\Pi_n = \mathrm{Conv}(w \cdot (0,1,\dots,n) \mid w \in \mathfrak{S}_{n+1}) \subset \mathbb{R}^{n+1}$. Let $E = \{0,1,\dots,n\}$.

$= \sum_{0 \leq i < j \leq n} \mathrm{Conv}(e_i, e_j)$ (a zonotope w/ edges being (positive) type A_n roots)

$\Sigma_{\Pi_n} = \widetilde{\Sigma}_{A_n}$ has cones: $\mathrm{Cone}(e_{S_1}, \dots, e_{S_k}) + \mathbb{R}\mathbb{1}$ for $\emptyset \subsetneq S_1 \subsetneq \cdots \subsetneq S_k \subsetneq E$.

$\mathrel{\rotatebox{90}{$\curvearrowleft$}}$ $\rightsquigarrow$ via a sequence of stellar subdivisions

Σ_{Δ_n}



Let $z_S = D_{e_S}$ for $\emptyset \subsetneq S \subsetneq E$, and $z_E = -\sum_{i \in S \subsetneq E} z_S$

Thm Let $P \subset \mathbb{R}^{n+1}$ be a lattice polytope. TFAE: (say P a <u>lattice generalized permutohedron</u>)

① $P \in \mathrm{Def}(\Pi_n)$ (equiv. $-P \in \mathrm{Def}(\Pi_n)$)

② Every edge of P is $/\!/$ to $e_i - e_j$ $\exists i \neq j$.

③ $P \cap \mathbb{Z}^{n+1}$ is a M-convex subset.

Thm For $D = \sum_{\emptyset \subsetneq S \subseteq E} \mathrm{rk}(S) z_S$ is nef iff $\mathrm{rk}: 2^E \to \mathbb{Z}$ is submodular with $\mathrm{rk}(\emptyset) = 0$,

in which case $P_D = \{m \in \mathbb{R}^{n+1} \mid \langle m, \mathbb{1}\rangle = -\mathrm{rk}(E) \text{ and } \langle m, e_S\rangle \geq -\mathrm{rk}(S)\ \forall S \subsetneq E\} \in \mathrm{Def}(\Pi_n)$

Rem [Aguiar-Ardila] Faces of gen. perm. are gen. perms that is also produce of smaller gen. perms $\rightsquigarrow$ Hopf monoid str.

[Pos '09] Permutohedra, Associahedra, and Beyond

Rem [Ardila-Castillo-E-Postnikov '20] $\rightsquigarrow$ Coxeter permutohedra

$\hookrightarrow$ in need of an upgrade! (Type A_n picture is really was GL_n story)

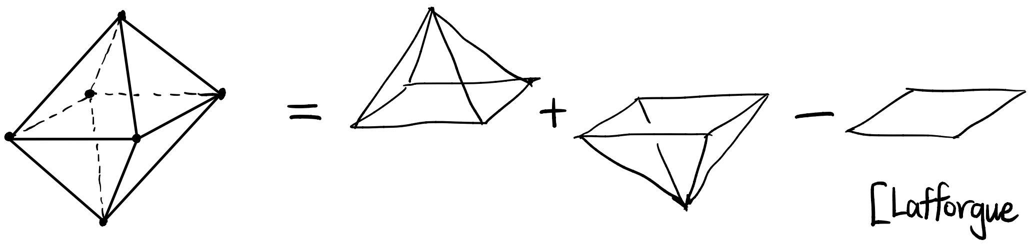
Let $L \subseteq \mathbb{C}^E$ realize a matroid M. $T = (\mathbb{C}^*)^E$ acts on $Gr(r;E)$.

Note that $\overline{T \cdot L} \simeq X_{P(M) \cap \mathbb{Z}^{n+1}}$ where $P(M) = \mathrm{Conv}(e_B \mid B \in \mathcal{B}(M))$.

$\simeq X_{\Sigma_{P(M)}}$ ($\because$ White)

<u>Thm</u> (GGMS '87) A lattice polytope in $[0,1]^{n+1}$ is the <u>base polytope</u> of a matroid iff it is a generalized permutohedron.

<u>E.g.</u> $P(U_{2,4})$:



[Lafforgue '03]

<u>Rem</u> If M conn. after removing loops & coloops, then $P(M)$ indeformable.

<u>Rem</u> (Gelfand–Serganova '87) Coxeter matroids (Borovik-Gelfand-White '03).

$\hookrightarrow$ also in need of an upgrade!

<u>Defn</u> A map $f : \{\text{matroids on } E\} \longrightarrow R$ is <u>valuative</u> if for every finite collection $a_i \in \mathbb{Z}$, M_i matroid on E st $\sum_i a_i 1_{P(M_i)} = 0$ where $1_S : \mathbb{R}^E \longrightarrow \mathbb{Z}$, $x \longmapsto \begin{cases} 1 & x \in S \\ 0 & \text{else} \end{cases}$

one has $\sum_i a_i f(M_i) = 0$.

<u>Prop</u> (Ardila-Fink-Rincón '10) $M \longmapsto T_M(x,y)$ is valuative.

<u>Thm</u> (Derksen–Fink '10) $\mathbb{Z}\{1_{P(M)} \mid M \text{ a matroid on } E\}$ has a basis given by Schubert matroids
(E.–Sanchez-Supina '21)

<u>Cor</u> Any additive relation of matroid invariants that holds for $\mathbb{C}$-linear matroids holds for arbitrary matroids if the invariants are valuative.

<u>Thm</u> (Fink-Speyer '12) Consider

$$\begin{tikzcd} & Fl(1,r,n;E) \arrow[dl, "\pi_r"'] \arrow[dr, "\pi_{1n}"] & \\ Gr(r;E) & & \mathbb{P}^n \times \mathbb{P}^n \end{tikzcd}$$

For $L \in Gr(r;E)$ realizing M, have $(\pi_{1n})_* \pi_r^* (\mathcal{O}_{\overline{T \cdot L}}(1)) = T_M(x,y)$

<u>Rem</u> (Dinu–E.–Seynnaeve '21) This for flag matroids. $\qquad \in \dfrac{\mathbb{Z}[x,y]}{\langle x^{n+1}, y^{n+1} \rangle} \simeq K(\mathbb{P}^n \times \mathbb{P}^n)$