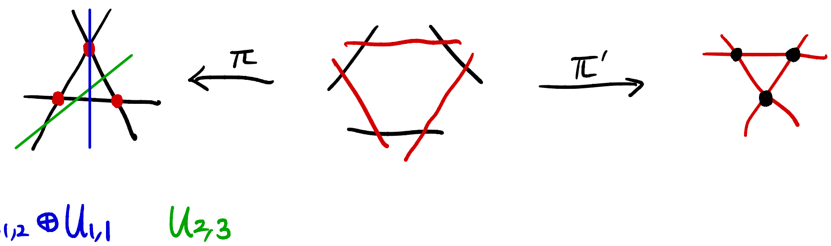
Lecture 10

E.g. $L = \mathbb{C}^E$, i.e. M is a Boolean matroid ($\mathcal{B} = \{E\}$). Then W_L in this case is known as a permutohedral variety X_{A_n}, built from $\mathbb{P}^n$ by blowing-up all (strict transforms of) coordinate subspaces in $\mathbb{P}^n$.

$E = \{0,1,2\}$:



$$\begin{tikzcd} & X_{A_2} \arrow[dl] \arrow[dr] & \\ \mathbb{P}^2 \arrow[rr, dashed, "\text{crem}"] & & \mathbb{P}^2 \end{tikzcd}$$

Rem $\overline{\mathcal{M}_{0,n}} = W_L^{\text{min'l}}$ where $M(L) = M(K_{n-1})$ and one uses min'l building set.

Prop When $L \subseteq \mathbb{C}^E$ realizes M, so we have $\mathbb{P}L \hookrightarrow \mathbb{P}^n$, the wnd cpt W_L is the closure of $\mathbb{P}L^\circ$ in X_{A_n}, i.e. the strict transform fitting into

$$\begin{tikzcd} W_L \simeq \widetilde{\mathbb{P}L} = \overline{\mathbb{P}L^\circ} \arrow[r, hook] \arrow[d] \arrow[dr] & X_{A_n} \arrow[d] \arrow[dr] & \\ \mathbb{P}L \arrow[r, hook] & \mathbb{P}^n \arrow[r, dashed, "\text{crem}"] & \mathbb{P}^n \\ & \mathbb{P}L^{-1} \arrow[ur, hook] & \end{tikzcd}$$

Defn Let α, β be pullbacks of source & target $\mathbb{P}^n$ hyperplane classes. Then:

Cor $\beta^{r-1} \cap [W_L] = \int_{X_{A_n}} \beta^{r-1} \cdot \eta_{W_L} = |\overline{\chi}_M(0)|$. Moreover, $\alpha^{r-1-i}\beta^i \cap [W_L] = \mu^i(M)$,
where $\overline{\chi}_M(q) = \mu^0(M)q^{r-1} - \mu^1(M)q^{r-2} + \cdots$

Cor $(\mu^i(M))_i$ form a log-conc. nonneg. seq. w/ no internal zeroes if M realizable.

pf) Pullbacks of b.p.f. are b.p.f., so α, β restricted to W_L are nef.
Now apply Khovanskii-Teissier to above Cor.

How to get arbitrary, not necessarily realizable, case?

Defn/Thm [Feichtner-Yuzvinsky '04] [de Concini-Procesi '95] The Chow ring of a loopless matroid M is

$$A^\bullet(M) := \frac{\mathbb{Z}[x_F \mid \emptyset \subsetneq F \subsetneq E \text{ flat in } M]}{\langle x_F x_{F'} \mid F, F' \text{ incomp.}\rangle + \langle \sum_{F \ni i} x_F - \sum_{G \ni j} x_G \mid i,j \in E \rangle}$$

and $A^\bullet(M) \simeq A^\bullet(W_L)$ via $x_F \mapsto$ exceptional divisor from blowing-up $\mathbb{P}L_F$.

N.B. Here, $\alpha_M = \sum_{F \ni i} x_F$, $\beta_M = \sum_{G \not\ni j} x_G$. $\int_M \alpha_M^{r-1} = 1$.

Thm [Adiprasito–Huh–Katz '18] $A^\bullet(M)_{\mathbb{R}}$ is a Poincaré duality algebra with $\int_M \alpha_M^{r-1} = 1$, and satisfies (mixed) $HR^{\leq 1}$ w/r/t the submodular cone

$$K_M = \{ \sum_F c_F x_F \mid c_{(\cdot)} : 2^E \to \mathbb{R} \text{ strictly submodular with } c_\emptyset = c_E = 0 \}.$$

In fact, $(A^\bullet(M)_{\mathbb{R}}, \int_M, K_M)$ has the Kähler package:

(PD) $\int_M : A^{r-1}(M) \xrightarrow{\simeq} \mathbb{Z}$ and $A^{r-1-i}(M) \times A^i \to \mathbb{Z}$ perfect.

(HL) For $\ell \in K_M$, $A^i(M) \xrightarrow{\times \ell^{r-1-2i}} A^{r-1-i}(M)$ is isom. $\forall i$.

(HR) For $\ell \in K_M$, $Q_\ell^i : A^i \times A^i \to \mathbb{Z}$, $(x, y) \mapsto (-1)^i \int_M x y \ell^{r-1-2i}$ is positive definite on $\ker(\ell^{r-2i})$.

Cor Resolution of [Heron–Rota–Welsh '70s], i.e. $\mu^i(M)$ log-conc. nonneg. seq. w/ no internal zeroes for arbitrary loopless matroid M.

Rem Proof of [AHK'18] is a double induction: rank M & order filters

rank M: for a flat F, $\mathcal{F}(M|_F) = [\hat{0}, F]$, $\mathcal{F}(M/F) = [F, \hat{1}]$.

$$\rightsquigarrow A^\bullet(M)/\operatorname{ann}(x_F) \simeq A^\bullet(M|_F) \otimes A^\bullet(M/F)$$

"= restricting to subvar. representing x_F"

$$f \in \mathbb{R}[w_0, \ldots, w_n] \longleftrightarrow A_f^\bullet = \mathbb{R}[\partial_0, \ldots, \partial_n]/\sim$$

$$\frac{\partial}{\partial w_i} f \longleftrightarrow A_f^\bullet / \operatorname{ann}(\partial_i)$$

Rem [Backman–E.–Simpson] $h_F := \alpha_M - \sum_{G \supseteq F} x_G$. Then,

$$A^\bullet(M)/\operatorname{ann}(h_F) \simeq A^\bullet(Tr_F(M)). \qquad (Tr(M) = Tr_F(M))$$

+ Lorentzian polynom. applied to h_F's (which are nef, whereas x_F were not).

[Brändén–Leake '21] Just polynomial proof.

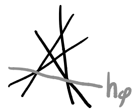
→ KEY in both: surface case (rank 3 case) $HR^{\leq 1}$ easy to show.

Rem Volume polynom.s : [Eur '20] [BES] [Dastidar–Ross]

Ques. Let $L_1 \subsetneq \cdots \subsetneq L_k$ be a flag of linear spaces.

Is $|\overline{\chi}_{M(L_i)}(0)|$ log-conc. with no internal zeroes?

(The original case $0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_k = L$ where $L_i = L \cap$ (general linear subsp.)).

(

$U_{1,2} \oplus U_{1,2} \overset{\varphi}{\twoheadleftarrow} U_{3,4}$

$H^0(\mathcal{O}(\tilde{H}-E_1-E_2)) = 1$, not nef.)

cf. [Eur-Huh '20] "# indep. subsets" still (ultra-)log-concave.