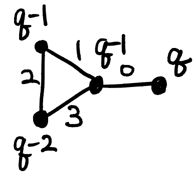
Lecture 1.

Syllabus (sheet sign)

Matroid $\quad M=(E,\mathcal{B}) \quad E=\{0,1,\dots,n\}\ , \quad \mathcal{B}\subset 2^E$

<u>Graph</u> G



$E=\{0,1,2,3\}\ , \quad \mathcal{B}=\{012,\ 013,\ 023\}$

chromatic polynom.

$\chi_G(q)=\#$proper colorings w/ $\leq q$ colors

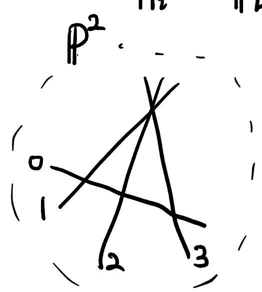
$= q(q-1)(q^2-3q+2)$

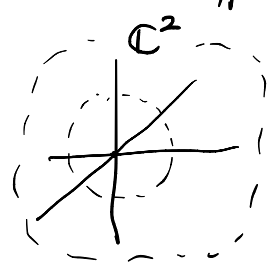
<u>Vectors</u> $\quad \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \end{array} = A \quad E=\{0,1,2,3\}\ , \quad \mathcal{B}=\{012,\ 013,\ 023\}$

$\updownarrow$

<u>Hyperplane arr.</u> (complement) $\quad L=\text{rowspan}(A)$. Have $\mathbb{P}^2\simeq \mathbb{P}L=\{x+y+z=0\}\subset \mathbb{P}^3 = \{[w:x:y:z]\}$

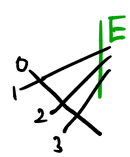
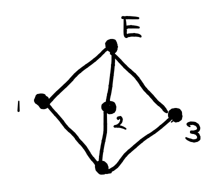
$H_i=\mathbb{P}L\cap i^{th}$ coord. hyperplane of $\mathbb{P}^3$



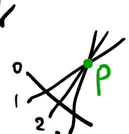


(a) $\mathbb{P}^2\setminus(\bigcup_i H_i)\simeq \mathbb{C}^2\setminus(H_1\cup H_2\cup H_3)\simeq \mathbb{C}^*\times(\mathbb{P}^1\setminus 3\text{pts})$

$P_X(q)=\sum_{m\geq 0}\dim H^i(X,\mathbb{Q})\,q^i \rightsquigarrow P_{\mathbb{P}^2\setminus\bigcup_i H_i}(q)=(1+q)(1+2q)=1+3q+2q^2$

(b) $(Bl_p\mathbb{P}^2,$  $)$ Boundary cplx:  top Betti $=\underline{2}$

$\downarrow \pi$

$(\mathbb{P}^2,$  $)$

(c) Let $\tilde{H}=\pi^*(\text{hyperplane class in }\mathbb{P}^2)=\pi^* c_1(\mathcal{O}_{\mathbb{P}^2}(1))$

$$h^0(K_{Bl_p\mathbb{P}^2}+\partial)=h^0(-3\tilde{H}+E+\tilde{H}+3(\tilde{H}-E)+E)=h^0(\tilde{H}-E)=\underline{2}$$

$\updownarrow$

lines thru p

(d) $\mathbb{P}^3 \xrightarrow{crem} \mathbb{P}^3 \quad [w:x:y:z] \longmapsto [\frac{1}{w}, \frac{1}{x}, \frac{1}{y}, \frac{1}{z}]$

$$\begin{array}{lll}
\mathbb{P}^3 & \xrightarrow{crem} & \mathbb{P}^3 \\
\cup & & \cup \\
\mathbb{P}L & \dashrightarrow & \{xy+yz+zx=0\} \qquad \deg = \underline{2} \\
\cup & & \\
\mathbb{P}L \cap H_{gen.} & \dashrightarrow & \text{twisted cubic curve} \qquad \deg = \underline{3} \\
\cup & & \\
pt & \dashrightarrow & pt \qquad \deg = \underline{1}
\end{array}$$

Conj. (Rota) |Coeff's| of $\chi_G(q)$ form a log-concave seq.
i.e. $(a_0, \cdots, a_d)$ $a_i > 0$ st $a_i^2 \geq a_{i-1}a_{i+1}$

Thm (Huh '12) Such coeff. are intersection degrees, hence log-conc. by Khovanskii-Teissier ineq.

Goal: Explain (a) ~ (d)

$$\begin{tikzcd}
\begin{pmatrix}\text{Vect. config.}\\ \text{Graphs}\end{pmatrix} \arrow[r, dash] \arrow[d, dash] & (\text{Alg. Geom.}) \arrow[d, dash] \\
(\text{matroids}) \arrow[r, dash] & (\text{Tropical/toric geom.})
\end{tikzcd}$$

Defn A matroid M on a finite ground set E is a nonempty collection $\mathcal{B}$ of subsets of E (called bases of M) such that for any $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \setminus B_2$, there exists $y \in B_2 \setminus B_1$ such that $B_1 \cup y - x \in \mathcal{B}$

Exer Bases of a matroid have same cardinality, called rank r of M.

E.g. (1) G finite graph $\rightsquigarrow$ $M(G)$ "cyclic matroid of G"
E = edges of G
$\mathcal{B}$ = {max'l acyclic subsets of edges}

(2) Linear matroids: $E = (v_0, \dots, v_n)$ vectors spanning a r-dim'l k-vec.sp. $L^\vee$.

$\longleftrightarrow (k^E \twoheadrightarrow L^\vee) \longleftrightarrow (L \subseteq k^E)$

$\mathcal{B} = \{\text{linear bases}\}$

"realizable over k"

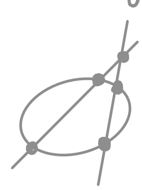
(3) Uniform matroids $U_{r,E}$ has $\mathcal{B} = \binom{E}{r}$.

$U_{2,4}$ realizable as 4-general vectors in k^2.
(note that k cannot be $\mathbb{F}_2$)

<u>Question</u>: For fixed $k = \mathbb{F}_q$, $q = p^m$, and fixed $r \geq 0$, what is max N such that $U_{r,N}$ is realizable?
(MDS max dist. separable conj.)

<u>Rem</u> For $r=3$, have $N = q+1$ if char > 2, $q+2$ if char $= 0$.

$\mathbb{P}^1_k \hookrightarrow \mathbb{P}^2_k$ conic



Corrado ≠ Beniamino [Segre'55]: every oval ($q+1$ gen position pts) arise in this way when char > 2

<u>Rem</u> $L \subseteq k^E \rightsquigarrow L \in Gr(r; E)(k) \hookrightarrow \mathbb{P}^{\binom{E}{r}-1}$

$r\,[\;\overset{|E|}{A}\;]$ rowspan $\longmapsto$ $r \times r$ minors.

Nonvanishing minors $\leftrightarrow$ bases of $M(L)$

Plücker coord. $\{P_I \mid I \in \binom{E}{r}\}$. For $j \in J \setminus I$, Plücker relation:

$$P_I P_J = \sum_{i \in I \setminus J} \pm P_{I-i\cup j} P_{J-j\cup i}$$

(These carve out Gr <u>set-theoretically</u>) but not as schemes if char > 0.

$\rightsquigarrow$ strong exchange properties

<u>Defn</u> $\mathbb{K}$ Krasner hyperfield $= \{0,1\}$ with $\boxplus$ & $\odot$ ↳ "usual" mult.
think of as "nonzero"
↳ set-valued:

$\boxplus$	0	1
0	$\{0\}$	$\{1\}$
1	$\{1\}$	$\{0,1\}$

(Matroids of rank r on E)

$\longleftrightarrow (P_I) \in \mathbb{K}^{\binom{E}{r}}$ such that Plücker rel. are satisfied,

i.e. $P_I P_J \boxplus \underset{i \in I \setminus J}{\boxplus} P_{I-i\cup j} P_{J-j\cup i}$ <u>contains 0</u>

$\forall\, I, J \in \binom{E}{r}$, $j \in J \setminus I$.