HODGE-RIEMANN RELATIONS AND LORENTZIAN POLYNOMIALS

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ABSTRACT. We introduce the theory of Lorentzian polynomials, motivated by the log-concavity properties that volume polynomials of projective varieties satisfy due to the validity of Hodge-Riemann relations.

1. HODGE-RIEMANN RELATIONS

Let $A^\bullet := \bigoplus_{i=0}^d A^i$ be a finitely generated, commutative, and graded $\mathbb{R}$-algebra, finite dimensional as an $\mathbb{R}$-vector space. For a linear map $\deg : A^d \to \mathbb{R}$, the pair $(A^\bullet, \deg)$ is said to satisfy *Poincare duality* if

(1) the *degree map* $\deg$ is an isomorphism, and
(2) for every $i = 0, \dots, d$, the pairing $A^i \times A^{d-i} \to \mathbb{R}$ defined by $(\alpha, \beta) \mapsto \deg(\alpha \cdot \beta)$ is non-degenerate.

Examples of such $A^\bullet$ with Poincare duality include

– $A^\bullet = H^{2\bullet}(X; \mathbb{R}) = \bigoplus_{i=0}^d H^{2i}(X; \mathbb{R})$, the cohomology ring in even degrees of a compact orientable manifold X of real dimension $2d$,
– $A^\bullet = H^{\bullet,\bullet}(X; \mathbb{R}) = \bigoplus_{i=0}^d H^{i,i}(X; \mathbb{R})$, the cohomology ring of real (i, i)-forms on a compact Kähler manifold X of complex dimension d, and in particular,
– $A^\bullet = A^\bullet(X)_\mathbb{R} = \bigoplus_{i=0}^d A^i(X)_\mathbb{R}$, the Chow ring (tensored by $\mathbb{R}$) of a smooth complex projective variety X of dimension d that admits an affine stratification—for instance, if X is a smooth projective toric variety or a generalized flag variety.

In the latter two cases listed above, the ring $A^\bullet$ satisfies further properties often referred to as "Lefschetz properties" or the "Kähler package." Let us focus on a particular portion:

Definition 1.1. Let $(A^\bullet, \deg)$ be a Poincare duality $\mathbb{R}$-algebra with $d \geq 2$. An element $\ell \in A^1$ is said to satisfy the *Hodge-Riemann relations in degrees* $\leq \mathbf{1}$, abbreviated (HR$^{\leq 1}$), if

(HR^0) $\deg(\ell^d) > 0$, and
(HR^1) the symmetric bilinear pairing $A^1 \times A^1 \to \mathbb{R}$ defined by $(\alpha, \beta) \mapsto \deg(\alpha \cdot \beta \cdot \ell^{d-2})$ is non-degenerate with exactly one positive eigenvalue.

For a subset $\mathscr{K}$ in A^1, a triple $(A^\bullet, \deg, \mathscr{K})$ is said to satisfy (HR$^{\leq 1}$) if every $\ell \in \mathscr{K}$ satisfies (HR$^{\leq 1}$). The triple $(A^\bullet, \deg, \mathscr{K})$ is said to satisfy the *mixed* (HR$^{\leq 1}$) if the conditions (HR^0) and (HR^1) are satisfied with ℓ^d and ℓ^{d-2} replaced by the products $\ell_1 \cdots \ell_d$ and $\ell_1 \cdots \ell_{d-2}$ for any choice of $\ell_1, \dots, \ell_d \in \mathscr{K}$.

We recall some classical results for a smooth complex projective variety X of dimension d. See [Laz04, §1.4 & §3.1] for details. The Lefschetz (1,1)-theorem states that $H^{1,1}(X; \mathbb{R})$ equals the image of the map $c_1 : (\operatorname{Pic} X)_\mathbb{R} \to H^2(X; \mathbb{R})$, so we will refer to the elements of $H^{1,1}(X; \mathbb{R})$ as divisor classes. The degree map $\deg : H^{d,d}(X; \mathbb{R}) \to \mathbb{R}$ making $H^{\bullet,\bullet}(X; \mathbb{R})$ into a Poincare duality algebra agrees with intersection theory: For divisors $D_1, \dots, D_d$, their intersection multiplicity as elements in the Chow ring of X is equal to $\deg(c_1(D_1) \cdots c_1(D_d))$. Let $\mathscr{K}(X) \subset H^{1,1}(X; \mathbb{R})$ be the *ample cone* of X, which is the cone generated by the classes of ample divisors on X, and let $\mathscr{K}(X)_\mathbb{Q}$ the set of classes of ample $\mathbb{Q}$-divisors on X.

Theorem 1.2. The triple $(H^{\bullet,\bullet}(X;\mathbb{R}), \deg, \mathscr{K}(X)_{\mathbb{Q}})$ satisfies mixed (HR$^{\leq 1}$).

Proof. Taking multiples if necessary, we may assume that $\ell_1, \ldots, \ell_d$ are classes of very ample divisors $D_1, \ldots, D_d$. Then, by Bertini's theorem, we may assume that the intersection $D_1 \cap \cdots \cap D_d$ is a collection of nonsingular points, and that $D_1 \cap \cdots \cap D_{d-2}$ is a smooth projective surface Y. Hence, $\deg(\ell_1 \cdots \ell_d) = \#(D_1 \cap \cdots \cap D_d) > 0$, proving mixed (HR^0). Writing $\alpha|_Y$ for the restriction of a divisor α on X to Y, we have $\deg(\alpha \cdot \beta \cdot \ell_1 \cdots \ell_{d-2}) = \deg(\alpha|_Y \cdot \beta|_Y)$. Repeated application of the Lefschetz hyperplane theorem implies that the restriction map $H^{1,1}(X;\mathbb{R}) \to H^{1,1}(Y;\mathbb{R})$ defined by $(\cdot) \mapsto (\cdot)|_Y$ is injective. Thus, the symmetric bilinear form $(\alpha, \beta) \mapsto \deg(\alpha \cdot \beta \cdot \ell_1 \cdots \ell_{d-1})$ is a restriction of the symmetric bilinear form on $H^{1,1}(Y;\mathbb{R})$ defined by $(\alpha, \beta) \mapsto \deg(\alpha \cdot \beta)$. The latter bilinear form (before restricting) is non-degenerate with exactly one positive eigenvalue by the Hodge index theorem for surfaces [Har77, V.1.9]. Thus, noting that restrictions of ample divisors are ample, we conclude by Cauchy's interlacing theorem (Lemma 2.2.(a)) that the original symmetric bilinear form on $H^{1,1}(X;\mathbb{R})$ is non-degenerate with exactly one positive eigenvalue, verifying mixed (HR^1) as desired. □

A key lesson from the above proof of Theorem 1.2 is that, after applying the classical Bertini theorem and Lefschetz hyperplane theorem to reduce dimension, the crux of the Hodge-Riemann relations in degrees ≤ 1, i.e. its the signature conditions, amounts to the Hodge index theorem for surfaces. In **Exercise 1**, you are asked to verify explicitly (HR$^{\leq 1}$) for cohomology rings of some smooth complex projective varieties.

Remark 1.3. An element $\ell \in A^1$ is said to satisfy the *hard Lefschetz properties* (in degree ≤ 1) if $\ell^d \neq 0$ and the symmetric bilinear pairing is non-degenerate. In other words, the "new" content of (HR$^{\leq 1}$) for rings with hard Lefschetz properties is the imposed signature properties. Theorem 1.2 combined with the validity of hard Lefschetz properties for $H^{\bullet,\bullet}(X;\mathbb{R})$ implies mixed (HR$^{\leq 1}$) for the triple $(H^{\bullet,\bullet}(X;\mathbb{R}), \deg, \mathscr{K}(X))$.

2. Log-concavity

A nonnegative sequence $(c_0, c_1, \ldots, c_d)$ is *log-concave* if $c_i^2 \geq c_{i-1}c_{i+1}$ for all $i = 1, \ldots, d-1$, and has *no internal zeros* if $c_i c_j > 0 \implies c_k > 0$ for all $0 \leq i < k < j \leq d$. A real-valued function $f : \mathbb{R}^n \to \mathbb{R}$ is *log-concave* at a point $w \in \mathbb{R}^n$ if $f(w) > 0$ and the Hessian matrix of $\log f$ is negative semidefinite at w. The (HR$^{\leq 1}$) property leads to two related log-concave properties.

Proposition 2.1. Let a triple $(A^\bullet, \deg, \mathscr{K})$ satisfy mixed (HR$^{\leq 1}$).

(a) For two elements α and β in the closure of $\mathscr{K}$, the sequence $(c_0, c_1, \ldots, c_d)$ defined by $c_i = \deg(\alpha^{d-i}\beta^i)$ is log-concave with no internal zeros.

(b) For elements $\eta_1, \ldots, \eta_n$ in A^1, let us define a polynomial $f \in \mathbb{R}[w_1, \ldots, w_n]$, called the *volume polynomial with respect to* $\eta_1, \ldots, \eta_n$, by

$$f(w_1, \ldots, w_n) = \frac{1}{d!} \deg\left((\eta_1 w_1 + \cdots + \eta_n w_n)^d\right)$$
$$= \sum_{\substack{a_1+\cdots+a_n=d \\ a_1,\ldots,a_n \geq 0}} \deg(\eta_1^{a_1} \cdots \eta_n^{a_n}) \frac{w_1^{a_1} \cdots w_n^{a_n}}{a_1! \cdots a_n!}.$$

If $\eta_1, \ldots, \eta_n$ are in the closure of $\mathscr{K}$, then f has nonnegative coefficients, and as a real-valued function $\mathbb{R}^n \to \mathbb{R}$, it is either identically zero, or is log-concave on the positive orthant $\mathbb{R}^n_{>0}$.

When the triple is $(H^{\bullet,\bullet}(X;\mathbb{R}), \deg, \mathscr{K}(X))$ for a smooth projective variety X, Proposition 2.1.(a) is known classically as Khovanskii-Teissier inequalities. The volume polynomial measures the volumes of Newton-Okounkov bodies associated to (nef) divisors [LM09, KK12]. As such, its log-concavity properties are closely related to the classical Brunn-Minkowski inequalities and Alexandrov-Fenchel inequalities of volumes of convex bodies. See [Laz04, §1.6] for a survey of these results.

Proof. One can verify that a limit of log-concave positive sequences are log-concave with no internal zeros. Thus, the conditions to be met in both (a) and (b) are closed conditions, so we may assume $\alpha, \beta, \eta_1, \dots, \eta_n$ to be elements of $\mathscr{K}$ (not just the closure of $\mathscr{K}$). In this case, both statements follow from Cauchy's interlacing theorem and related results: For (a), setting $\gamma_i = \alpha^{d-i-1}\beta^{i-1}$ for each $i = 1, \dots, d-1$, we have that the matrix $\begin{bmatrix} \deg(\alpha^2\gamma_i) & \deg(\alpha\beta\gamma_i) \\ \deg(\alpha\beta\gamma_i) & \deg(\beta^2\gamma_i) \end{bmatrix}$ has a non-positive determinant by combining mixed (HR$^{\leq 1}$) with Lemma 2.2.(a). For (b), the nonnegativity of coefficients follows from mixed (HR^0), and at every $u \in \mathbb{R}^n_{>0}$, consider the symmetric bilinear pairing in (HR^1) with $\ell = u_1\eta_1 + \cdots + u_n\eta_n$, which is non-degenerate with exactly one positive eigenvalue. The restriction to $\operatorname{span}(\eta_1, \dots, \eta_n)$ of this pairing also is non-degenerate with exactly one positive eigenvalue by Lemma 2.2.(a), and its matrix with respect to the spanning set $\eta_1, \dots, \eta_n$ is the Hessian matrix $\mathcal{H}_f(u)$ of f at u, which may be singular but still has exactly one positive eigenvalue. Now apply Lemma 2.3. □

A key lesson from the proof of Proposition 2.1 is that limiting arguments, which are quite useful, result in certain additional constraints (e.g. no internal zeros) that cannot be detected from looking only at the quadric case (i.e. the "surface case").

Next two lemmas, used in the proofs above, are variations on the theme of Cauchy's interlacing theorem.

Lemma 2.2. [Ser10, §6] For an $n \times n$ real symmetric (or more generally Hermitian) matrix A, set $\lambda_1(A) \leq \cdots \leq \lambda_n(A)$ to be the eigenvalues of A.

(a) If B is a principal $m \times m$ submatrix of A, then

$$\lambda_i(A) \leq \lambda_i(B) \leq \lambda_{n-m+i}(A)$$

for all $i = 1, \dots, m$. In particular, if $m = n-1$ then

$$\lambda_1(A) \leq \lambda_1(B) \leq \cdots \leq \lambda_{n-1}(A) \leq \lambda_{n-1}(B) \leq \lambda_n(A).$$

(b) For any $v \in \mathbb{R}^n$, let $B = A + vv^T$, a rank 1 modification of A. Then, one has

$$\lambda_1(A) \leq \lambda_1(B) \leq \cdots \leq \lambda_{n-1}(A) \leq \lambda_{n-1}(B) \leq \lambda_n(A) \leq \lambda_n(B).$$

One can use Lemma 2.2.(b) to show:

Lemma 2.3. [BH20, Proposition 2.33] A nonzero homogeneous polynomial f of degree ≥ 2 with nonnegative coefficients is log-concave on $\mathbb{R}^n_{>0}$ if and only if the Hessian matrix of f has exactly one positive eigenvalue on $\mathbb{R}^n_{>0}$.

3. LORENTZIAN POLYNOMIALS

For a homogeneous polynomial $f \in \mathbb{R}[w_1, \ldots, w_n]$ of degree $d \geq 2$, the graded $\mathbb{R}$-algebra

$$A_f^\bullet = \frac{\mathbb{R}[\partial_1, \ldots, \partial_n]}{\left\langle g \in \mathbb{R}[\partial_1, \ldots, \partial_n] \;\middle|\; g \cdot f = 0 \text{ where } \partial_i\text{'s act on } f \text{ by partial derivatives } \frac{\partial}{\partial w_i} \right\rangle}$$

is a Poincare duality algebra with $\deg : A_f^d \to \mathbb{R}$ given by $\deg(g) = g \cdot f / d!$. It is straightforward to check that f is the volume polynomial of $A_f^\bullet$ with respect to $\partial_1, \ldots, \partial_n$. In other words,

$$\left\{ \begin{matrix} \text{homogeneous polynomials} \\ \text{in } \mathbb{R}[w_1, \ldots, w_n] \end{matrix} \right\} \underset{A_f^\bullet}{\overset{\text{volume polynomial}}{\rightleftarrows}} \left\{ \begin{matrix} \text{graded quotient rings of } \mathbb{R}[\partial_1, \ldots, \partial_n] \\ \text{that are Poincare duality algebras} \end{matrix} \right\}.$$

For our key question, let us define $\mathscr{K}_f := \{a_1\partial_1 + \cdots + a_n\partial_n \mid a_1, \ldots, a_n > 0\} \subset A_f^1$.

Question 3.1. When does the triple $(A_f^\bullet, \deg, \mathscr{K}_f)$ satisfy mixed (HR$^{\leq 1}$)? That is, how do we extend the correspondence above when we further impose mixed (HR$^{\leq 1}$) on the right-hand-side?

Let H_n^d be the space of all homogeneous degree d polynomials in $\mathbb{R}[w_1, \ldots, w_n]$. The *support* of $f \in H_n^d$, denoted $\operatorname{supp}(f)$ is the set of exponent vectors of the monomials appearing in f with nonzero coefficients. We write $d\Delta_n = \{\boldsymbol{m} = (m_1, \ldots, m_n) \in \mathbb{Z}_{\geq 0}^n \mid m_1 + \cdots + m_n = d\}$. A subset $S \subseteq d\Delta_n$ is *M-convex* if for any $\boldsymbol{m}, \boldsymbol{m}' \in S$ and $i \in \{1, \ldots, n\}$ with $m_i > m_i'$, there exists $j \in \{1, \ldots, n\}$ such that $m_j < m_j'$ and $\boldsymbol{m} - e_i + e_j$ and $\boldsymbol{m}' - e_j + e_i \in S$. Equivalently, S is M-convex if it is the set of lattice points of a *generalized permutohedra*, which are polytopes characterized by each edge being parallel to $e_i - e_j$ for some $i, j \in \{1, \ldots, n\}$.

Theorem 3.2. For $d \geq 2$, the following four subsets of H_n^d coincide.

(I) The space of *Lorentzian polynomials*, defined as the closure in H_n^d of the subset

$$\mathring{L}_n^d = \left\{ f \in H_n^d \;\middle|\; \begin{array}{l} f \text{ has positive coefficients (with support } d\Delta_n\text{), and} \\ \text{for every choice of } i_1, \ldots, i_{d-2}, \text{ the quadratic form} \\ \partial_{i_1} \cdots \partial_{i_{d-2}} f \text{ has the Lorentzian signature } (+, -, \cdots, -) \end{array} \right\}.$$

(II) The set L_n^d defined as

$$L_n^d = \left\{ f \in H_n^d \;\middle|\; \begin{array}{l} f \text{ has nonnegative coefficients with M-convex support, and} \\ \text{for every choice of } i_1, \ldots, i_{d-2}, \text{ the quadratic form} \\ \partial_{i_1} \cdots \partial_{i_{d-2}} f \text{ has at most one positive eigenvalue} \end{array} \right\}.$$

(III) The space of *completely log-concave polynomials*, defined as

$$\left\{ f \in H_n^d \;\middle|\; \begin{array}{l} f \text{ has nonnegative coefficients, and for every nonnegative } m \times n \text{ matrix } (a_{ij}), \\ (a_{11}\partial_1 + \cdots + a_{1n}\partial_n) \cdots (a_{m1}\partial_1 + \cdots + a_{mn}\partial_n) \cdot f \\ \text{is either identically zero or log-concave on } \mathbb{R}_{>0}^n \end{array} \right\}.$$

(IV) The set of volume polynomials of Poincare duality algebras satisfying mixed (HR$^{\leq 1}$), that is, the set

$$\left\{ f \in H_n^d \;\middle|\; \text{the triple } (A_f^\bullet, \deg, \mathscr{K}_f) \text{ satisfies mixed (HR}^{\leq 1}) \right\}.$$

Proof. The equivalence of (I) and (II) is [BH20, Theorem 2.25], and of (II) and (III) is [BH20, Theorem 2.30]. The inclusion (IV) $\subseteq$ (I) follows from a minor modification of the proof of Proposition 2.1.(b), which yields that f in (IV) is a limit of polynomials in $\mathring{L}^d_n$. For the inclusion (III) $\subseteq$ (IV), checking mixed (HR^0) is immediate. To check mixed (HR^1), set $m = n - 2$ and fix an arbitrary strictly positive matrix (a_{ij}). The resulting symmetric form of the quadric obtained from taking the directional derivatives has exactly one positive eigenvalue, and each zero eigenvalue gives a relation that belongs to the set of linear relations defining A^1_f as a quotient of the degree 1 part of $\mathbb{R}[\partial_1, \dots, \partial_n]$. Thus, the symmetric pairing to be checked for (HR^1) is non-degenerate with exactly one positive eigenvalue. □

In **Exercise 2**, you are asked to verify that the converses of the following implications are false.

$$f \text{ is Lorentzian} \implies \text{the triple } (A^\bullet_f, \deg, \mathscr{K}_f) \text{ satisfies (HR}^{\leq 1}) \implies f \text{ is log-concave on } \mathbb{R}^n_{>0}.$$

Lorentzian polynomials generalize volume polynomials of projective varieties in the following way.

Corollary 3.3. [BH20, Theorem 4.6] Let $\eta_1, \dots, \eta_n$ be elements in the closure of the ample cone of a (not necessarily smooth) projective variety over an algebraically closed field (of arbitrary characteristic). Then, the corresponding volume polynomial is Lorentzian.

In particular, the support of such volume polynomial is M-convex, which generalizes [CCRL+20, Proposition 5.4]. You are asked to prove Corollary 3.3 in **Exercise 3**. Finding Lorentzian polynomials that do not arise as a volume polynomial as given in Corollary 3.3 is an open problem [BH20, Question 4.9]. We remark that Lorentzian polynomials also generalize *stable polynomials* or *hyperbolic polynomials*, which are multivariate generalizations of real-rooted univariate polynomials; see [Wag11] for a survey. The closely related concept of completely log-concave polynomials appeared in [AOGV18].

Lorentzian polynomials are preserved various operations [BH20, §3]. We only mention one kind here, nonnegative linear tranformations, stated below. For a characterization of linear operators on the space of polynomials that preserve Lorentzian polynomials, see [BH20, Theorem 3.2].

Theorem 3.4. [BH20, Theorem 2.10] Let $f \in H^d_n$ be a Lorentzian polynomial, and A a nonnegative $n \times m$ matrix. Then $f(Av) \in H^d_m \subset \mathbb{R}[v_1, \dots, v_m]$ is a Lorentzian polynomial.

We now list some applications of Lorentzian polynomials to combinatorics. While many of the applications are in matroid theory, we speculate that this is more a reflection of the fact that matroid theory happened to be a research focus of the developers of Lorentzian polynomials.

- In [BH18] (reproduced in [BH20, §4.3]) and [ALOGV18], the authors showed that the multivariate Tutte polynomial of a matroid M is Lorentzian, and consequently deduced Mason's conjecture on the ultra-log-concavity of the number of independent subsets of varying sizes.
- The authors of [EH20] generalized the previous item to morphisms of matroids.
- The authors of [BES19] showed that the volume polynomial with respect to a certain set of generators for the Chow ring a matroid is Lorentzian, and consequently deduced a simplified proof of the Hodge theory of matroids in degrees ≤ 1, developed originally in [AHK18].
- The authors of [HMMSD19] showed that normalized Schur polynomials are Lorentzian, and consequently deduced a special case of Okounkov's conjecture on the Littlewood-Richardson coefficients.
- The authors of [BLP20] used the theory of Lorentzian polynomials to give new lower bounds on the volumes of flow and transportation polytopes.

4. EXERCISES

Exercise 1. Getting familiar with (HR$^{\leq 1}$) on projective varieties

For each smooth projective variety X below, explicitly verify (HR$^{\leq 1}$) for the triple $(R^\bullet, \deg, \mathscr{K}(X))$.

(a) $X = \mathbb{P}^2 \times \mathbb{P}^2$.
(b) $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

A boundary element in the closure of $\mathscr{K}(X)$ often fails to satisfy (HR$^{\leq 1}$) in several ways. Let $X = \mathrm{Bl}_p \mathbb{P}^3$, the blow-up of $\mathbb{P}^3$ at a point, which is the closure in $\mathbb{P}^3 \times \mathbb{P}^2$ of the rational map $\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ given as the projection from the point p. Its ample cone is a 2-dimensional cone with two boundary rays, corresponding to the two distinguished maps $X \to \mathbb{P}^3$ and $X \to \mathbb{P}^2$.

(c) Show that one boundary ray gives a base-point-free divisor for which (HR^0) holds but (HR^1) fails, and that the other gives a base-point-free divisor for which (HR^0) fails but (HR^1) holds.

For those not familiar with cohomology rings of complex varieties and their ample cones, see the bottom of this page for the restatement of Exercise 1 in a purely algebraic language.

Exercise 2. Normalization and log-concave sequences

The *normalization* $N(f)$ of a polynomial f is obtained by replacing each monomial $w_1^{m_1} \cdots w_n^{m_n}$ appearing in f with $\frac{w_1^{m_1} \cdots w_n^{m_n}}{m_1! \cdots m_n!}$.

(a) Let f be a bivariate homogeneous polynomial $f = \sum_{i=0}^d c_i x^i y^{d-i} \in \mathbb{R}_{\geq 0}[x, y]$ of degree d with nonnegative coefficients. Show that

$$N(f) \text{ is Lorentzian} \iff (c_0, \ldots, c_d) \text{ is log-concave with no internal zeros.}$$

(b) Let $f = x^3y + xy^3$. Conclude from (a) that f is not Lorentzian. Verify that: f is log-concave on $\mathbb{R}^2_{>0}$, but the triple $(A^\bullet_f, \deg, \mathscr{K}_f)$ fails (HR$^{\leq 1}$).
(c) Let $f = x^3y + x^2y^2 + xy^3$. Conclude from (a) that f is not Lorentzian. Verify that: The triple $(A^\bullet_f, \deg, \mathscr{K}_f)$ satisfies (HR$^{\leq 1}$), so f is log-concave on $\mathbb{R}^2_{>0}$, but the triple fails mixed (HR$^{\leq 1}$).

To go beyond bivariate polynomials, let us define the following notion. For a homogeneous polynomial $f \in \mathbb{R}[w_1, \ldots, w_n]$ of degree d with nonnegative coefficients, we say that its coefficients form a *log-concave simplex* if, for any $1 \leq i < j \leq n$ and a monomial $\boldsymbol{w}^{\boldsymbol{m}}$ of degree $d' \leq d$, the coefficients of $\{w_i^k w_j^{d-d'-k} \boldsymbol{w}^{\boldsymbol{m}}\}_{0 \leq k \leq d-d'}$ in f form a log-concave sequence.

(d) Show that if $N(f)$ is Lorentzian, then the coefficients of f form a log-concave simplex.
(e) In contrast to the bivariate case, give an example of a polynomial f whose support is M-convex and whose coefficients form a log-concave simplex, but $N(f)$ is not Lorentzian.

Exercise 3. Volume polynomials are Lorentzian

Prove Corollary 3.3 by adapting the proof of Theorem 1.2 and the proof of Proposition 2.1.(b). You may use resolution of singularities for surfaces (which is valid in arbitrary characteristic).

Exercise 1 without geometry

(a) The ring is $A^\bullet = \mathbb{R}[x, y]/\langle x^3, y^3 \rangle$ with $\deg : x^2y^2 \mapsto 1$ and $\mathscr{K} = \{ax + by \mid a, b > 0\}$.
(b) The ring is $A^\bullet = \mathbb{R}[x, y, z]/\langle x^2, y^2, z^2 \rangle$ with $\deg : xyz \mapsto 1$ and $\mathscr{K} = \{ax + by + cz \mid a, b, c > 0\}$.
(c) The ring is $A^\bullet = \mathbb{R}[h, e]/\langle he, h^3 - e^3 \rangle$ with $\deg : h^3 \mapsto 1$ and $\mathscr{K} = \{ah + b(h - e) \mid a, b > 0\}$.

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