# 1 A grab bag?

- AM-GM: Given $x_1, ..., x_n$, $\sum x_i/n \leq (\Pi x_i)^{1/n}$.
- Weighted AM-GM: Given $a_1, ..., a_n$, $i_1, ..., i_n$ weighted such that $\sum i_j = 1$, we have $\sum a_j i_j \geq \prod a_j^{i_j}$.
- Cauchy-Schwarz: Given two vectors $u, v$, $|u \cdot v|^2 \leq ||u||^2 ||v||^2$. Alternatively, given $x_1, ..., x_n$ and $y_1, ..., y_n$, you can find that $(\sum x_i y_i)^2 \leq (\sum x_i^2)(\sum y_i^2)$. This can also be expressed as $(\sum \sqrt{x_i y_i})^2 \leq (\sum x_i)(\sum y_i)$, assuming $x_i, y_i \geq 0$. Square rooting both sides gives one final form $\sum \sqrt{x_i y_i} \leq \sqrt{(\sum x_i)(\sum y_i)}$
- Jensen: Let $f$ be a convex function, $t \in (0, 1)$. Then $tf(x_1) + (1-t)f(x_2) \geq f(tx_1 + (1-t)x_2)$.

# 2 Problems Part 1.5

A different set of problems which are on the easier...? end. Note: A bit proofy as well. Some problems do reuse parts of proofs from other problems.

1. Prove that $3(a^2 + b^2 + c^2) \geq (a + b + c)^2 \geq 3(ab + bc + ca)$.
2. Prove that $a^3 + b^3 + c^3 \geq a^2 b + b^2 c + c^2 a$, for $a, b, c > 0$.
3. Suppose $a, b, c > 0$ such that $a + b + c = 3$. Prove that $a^2 + b^2 + c^2 + ab + bc + ca \geq 6$.
4. Let $a, b, c > 0$ such that $abc = 1$. Prove that $a^2 + b^2 + c^2 \geq a + b + c$.
5. Let $a, b, c, d > 0$ and $a + b + c + d = 4$. Show that $\frac{4}{abcd} \geq \frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}$
6. Prove that $\sqrt{3x^2 + xy} + \sqrt{3y^2 + yz} + \sqrt{3z^2 + zx} \leq 2(x + y + z)$.
7. Suppose $a^2 + b^2 + c^2 + d^2 = 4$. Show that $\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{d} + \frac{d^2}{a} \geq 4$.

# 3 Problems Part 2.1

A different set of harder problems I didn't have time to get solutions for.

1. Let $a, b, c > 0$ such that $abc = 1$. Prove that
$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq a + b + c$$
2. Let $a, b, c > 0$ such that $a + b + c = 1$. Prove that $\frac{a}{\sqrt{a+2b}} + \frac{b}{\sqrt{b+2c}} + \frac{c}{\sqrt{c+2a}} < \sqrt{3/2}$.
3. Suppose $a, b, c, d \geq 0$. Prove that $a/(b + c) + b/(c + d) + c/(d + a) + d/(a + b) \geq 2$.
(Hint 1: There's more than one possible choice to multiply the LHS by.)
(Hint 2: You're going to need both AM-GM and C-S)
4. Let $a, b, c > 0$. Prove that $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + \frac{3(abc)^{1/3}}{a+b+c} \geq 4$.

# 4 Problems Part 2

Reposted from last week's part 2 (pretty difficult):

1. Let $x, y, z$ be distinct real numbers such that $x+y+z=0$. Find the maximum possible value of $\frac{xy+yz+xz}{x^2+y^2+z^2}$.
2. Let $x$ be a positive real number. Find the maximum value of $\frac{x^2+2-\sqrt{x^4+4}}{x}$.
3. For $n \geq 2$ let $a_1$, $a_2$, ..., $a_n$ be positive real numbers such that

$$(a_1 + a_2 + ... + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}\right) \leq \left(n + \frac{1}{2}\right)^2$$

   Prove that $\max(a_1, a_2, ..., a_n) \leq 4\min(a_1, a_2, ..., a_n)$.

4. Let $a$, $b$, $c$ be positive real numbers such that $a^2 + b^2 + c^2 + (a+b+c)^2 \leq 4$. Prove that

$$\frac{ab+1}{(a+b)^2} + \frac{bc+1}{(b+c)^2} + \frac{ca+1}{(c+a)^2} \geq 3.$$