Modular Arithmetic and Polynomials

Varsity Practice 7/26/20
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1 Vieta's

Given a polynomial like $a_n x^n + ... + a_1 x^1 + a_0 x^0$, we can always factor it into $(x - r_1)(x - r_2)...(x - r_n)$. Then, what Vieta's tells us to do is to look at a_{n-k}; this is the sum of all products of k different roots.

As an example, $x^3 + 2x^2 + 3x + 4$ has 3 roots x, y, z. These roots satisfy:

$$xyz = 4; xy + yz + xz = 3; x + y + z = 2$$

So whenever you get a polynomial, you can turn it into this series of equations. Solving for specifically x, y or z ends up giving you back your original polynomial.

2 Newton's

Given a polynomial like $a_n x^n + .. + a_1 x^1 + a_0 x^0$, suppose it has roots $r_1, ..., r_n$. Then let:

$$P_1 = r_1 + ... + r_k P_2 = r_1^2 + ... + r_k^2 ... P_k = r_1^k + ... + r_k^k$$

Then Newton's sums tells us that

$$a_n P_1 + a_{n-1} = 0; a_n P_2 + a_{n-1} P_1 + 2a_{n-2} = 0; a_n P_3 + a_{n-1} P_2 + a_{n-2} P_1 + 3a_{n-3} = 0; ...$$

where $a_j = 0$ for $j < 0$. Proof: Um... look it up on AoPS. It's pretty cool.

3 Problems

1. Suppose $x^2 + y^2 = 1$ and $x^4 + y^4 = \frac{17}{18}$. Find xy.
2. Let $p(x) = x^6 + 3x^5 - 3x^4 + ax^3 + bx^2 + cx + d$. Given that all the roots of this polynomial are either m or n (which are both integers), compute $p(2)$.
3. Suppose a polynomial $x^3 - x^2 + bx + c$ has roots a, b, c. What is the square of the minimum value of abc?
4. Let x, y, z be real numbers which sum to 0. Find the maximum value of:

$$\frac{xy + yz + zx}{x^2 + y^2 + z^2}$$

5. Karina starts with $p_1 = x^2 + x + k$. She notices that this has two integer roots $a_1 \geq b_1$. So she writes a new polynomial $p_2 = x^2 + a_1x + b_1$. She notices that this one also has integer roots a_2, b_2 so she writes the new polynomial $p_3 = x^2 + a_2x + b_2$. She continues doing this until she gets to p_7 and finds that it doesn't have integer roots. What's the largest possible value of k?

6. Let a, b, c be the roots of $x^3 - x + 1$. Find $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1}$.

7. Let $f(x) = 3x^3 - 5x^2 + 2x - 6$. If the roots are a, b, c, find:

$$(\frac{1}{a-2})^2 + (\frac{1}{b-2})^2 + (\frac{1}{c-2})^2$$

8. Let a, b, c be the roots of $x^3 - 9x^2 + 11x - 1$, and let $s = \sqrt{a} + \sqrt{b} + \sqrt{c}$. Find $s^4 - 12s^2 - 18s$.