21-832 PDE II   Fall 2016   R.L. Pego

Homework 3   Due Wednesday, Nov. 9

1. (Evans 5.21, $H^s(\mathbb{R}^n)$ is a Banach algebra for $s > n/2$) Show that whenever $s > n/2$, there is a constant C such that for all $u, v \in H^s(\mathbb{R}^n)$,
$$\|uv\|_{H^s} \le C\|u\|_{H^s}\|v\|_{H^s}.$$

2. On the vector space $H = \{u \in L^6(\mathbb{R}^3) : Du \in L^2(\mathbb{R}^3)\}$ define the inner product
$$(u, v) = \int_{\mathbb{R}^3} Du \cdot Dv.$$
(a) Prove that H is complete, thus a Hilbert space.

(b) Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary. Show that for any $v \in L^2(\partial\Omega)$ there is a unique $u \in H$ such that
$$\int_{\mathbb{R}^3} Du \cdot D\varphi = \int_{\partial\Omega} v\varphi \, dS$$
for all $\varphi \in H$.

(c) Suppose that v is continuous and the solution u in (b) happens to be piecewise C^1, with $u|_\Omega \in C^1(\overline{\Omega})$ and $u|_{\Omega^c} \in C^1(\Omega^c)$ where $\Omega^c = \mathbb{R}^3 \setminus \Omega$. Show that then u satisfies
$$\Delta u = 0 \quad \text{in } \mathbb{R}^3 \setminus \partial\Omega, \qquad [\nu \cdot Du] = v \quad \text{on } \partial\Omega.$$
Here, for $x \in \partial\Omega$, $[\nu \cdot f](x) = \lim_{t\to 0^+} \nu(x) \cdot (f(x + t\nu) - f(x - t\nu))$ denotes the normal jump across $\partial\Omega$, with $\nu(x)$ the outward unit normal.

(d) Suppose $\Omega = \cup_{j=1}^n B_j$ is a union of non-touching balls $B_j = B(x_j, R_j)$, and $v = v_j$ is constant on ∂B_j. For all $x \in \mathbb{R}^3$, find a formula for the solution $u(x)$ in (b), which has the form of a superposition of *monopoles* outside Ω:
$$u(x) = \sum_j \frac{a_j}{|x - x_j|}, \qquad x \notin \Omega.$$

3. Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain with C^1 boundary. Let $J : L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$ be the natural embedding, defined by
$$\langle Jf, v\rangle = \int_\Omega fv \qquad \text{for } v \in H_0^1(\Omega).$$
Prove that J is compact.

4. (A nonlinear boundary-value problem) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^1 boundary and let $a \in \mathbb{R}$. Write an appropriate *weak formulation* for the boundary value problem

$$-\Delta u + a \sin u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

Using a contraction-mapping argument, prove that for $|a|$ sufficiently small, there is a unique solution $u \in H_0^1(\Omega)$ to the weak formulation.

(Remark: The map $u \mapsto \sin(u)$ is not Frechet differentiable on $L^2(\Omega)$. Indeed, if $g : \mathbb{R} \to \mathbb{R}$ is smooth and the composition map $u \mapsto g \circ u$ is Frechet differentiable on $L^2(\Omega)$, then g must be affine!)