Recall: the row space of a matrix is the span of the rows of that matrix, denoted \( \mathcal{R}(A) \) for a matrix \( A \).

Q: Given a matrix, how do we find a basis of its row space?

A: We put \( A \) in (row) echelon form.

Why does this work?

Let \( U \) be the echelon form of \( A \). Then:

1. \( \mathcal{R}(A^T) = \mathcal{R}(U^T) \) because, as proved in class, row operations preserve the row space.
2. The non-zero rows of \( U \) span \( \mathcal{R}(U^T) \).
3. The non-zero rows of \( U \) are linearly independent, as proven in class (non-zero rows of matrices in echelon form are linearly independent).
It follows from (2) & (3) that the non-zero rows of $U$ form a basis of $C(U^T)$. Combining this with (1) tells us that they form a basis of $C(A^T)$.

**Example:** Find a basis of the row space of $A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 2 & 2 & 1 & 2 \end{pmatrix}$

\[
\begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 2 & 2 & 1 & 2 \end{pmatrix}
\rightarrow
\begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]

Therefore, a basis of $C(A^T)$ is $\{ (1, 2, 0, 1), (0, -2, 1, 0) \}$.

The method above does not work for column spaces.

Indeed, row operations do not preserve the column space so the column space of $A$ is not equal to the column space of its (row) echelon form, i.e. $C(A) \neq C(U)$.

In the example above, $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \in C(A)$ but $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \notin C(U)$. 
Completion to a basis.

Recall: any linearly independent set of vectors in a vector space $V$ can be completed to a basis of $V$.

For example: given $\vec{v}_1 = (1, 2, 3)$ and $\vec{v}_2 = (1, 1, 0)$, how do we complete this to a basis of $\mathbb{R}^3$?

$$\text{span } \{ \vec{v}_1, \vec{v}_2 \} = (A^T)$$ for $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 0 \end{pmatrix} \implies (1 2 3) (1 1 0) = 0$

i.e. $\text{span } \{ \vec{v}_1, \vec{v}_2 \} = (C(0^T))$.

Q: What is in $\mathbb{R}^3$ but linearly independent from all vectors in $(C(0^T))$?

A: $\vec{v}_3 = (0, 0, 1)$.

Then $\{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \}$ is a linearly independent set in $\mathbb{R}^3$.

Since the dimension of $\mathbb{R}^3$ is precisely 3, the set $\{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \}$ must actually be a basis for $\mathbb{R}^3$. 
Let $P_n(\mathbb{R})$ be the vector space of polynomials of degree at most $n$ over $\mathbb{R}$. Recall that $P_n(\mathbb{R})$ has dimension $n+1$.

Q: Let $W = \{ p \in P_5(\mathbb{R}) \mid p''' + p = 0 \}$
   
   (a) Prove that $W$ is a subspace of $P_5(\mathbb{R})$.
   
   (b) Find a basis of $W$.
   
   (c) Find the dimension of $W$.

Let $p \in W \subseteq P_5(\mathbb{R})$. We may write, for some $a_0, a_1, \ldots, a_5 \in \mathbb{R}$,

$$ p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 $$

for every $x \in \mathbb{R}$.

Then $p'''(x) = 6a_5 + 24a_4 x + 60a_5 x^2$ and hence

$$ (p''' + p)(x) = (a_0 + 6a_3) + (a_1 + 24a_4)x + (a_2 + 60a_5)x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5. $$

Since $p \in W$ it follows that

$$\begin{cases}
  a_0 + 6a_3 = 0 & a_3 = 0 \\
  a_1 + 24a_4 = 0 & a_4 = 0 \\
  a_2 + 60a_5 = 0 & a_5 = 0
\end{cases}$$
i.e. \( a_3 = a_4 = a_5 = 0 \), and hence \( a_0 = a_1 = a_2 = 0 \).

We deduce that \( q = 0 \), i.e. \( W = \{ 0 \} \).

Since \( \{ 0 \} \subseteq W \) because if \( p \) is the zero polynomial then
\[
p''' + q = 0 + 0 = 0,
\]
i.e. \( q \in W \).

So finally:\n\[
W = \{ 0 \}, \text{ hence } W \text{ has no basis and dim } W = 0.
\]