Number Theory

(a) First, we proceed by contradiction. Suppose that there are finitely many primes, namely \( p_1, p_2, \ldots, p_n \), where
\[ p_1 < p_2 < \ldots < p_n. \]

Observe that \( p_1 p_2 \cdots p_n + 1 \) is prime (since it is not divisible by any \( p_i \) for \( i = 1, \ldots, n \)). However, \( p_1 p_2 \cdots p_n + 1 > p_i \) for every \( i = 1, \ldots, n \) and so it is not amongst the set of prime numbers \( \{ p_1, \ldots, p_n \} \). This is a contradiction, so there are infinitely many primes.

(b) Second, we use the fact that if \( m \) and \( n \) are distinct non-negative integers then \( \gcd (2^m + 1, 2^n + 1) \) is finite. Let \( p_n \) be the least prime factor in \( 2^m + 1 \). Then, if \( i \neq j \) then \( p_i \neq p_j \) (otherwise \( \gcd (2^m + 1, 2^n + 1) = 1 + p_i - p_j + 1 \)). Therefore, the set \( \{ p_i \mid i \in \mathbb{N} \} \) is an infinite set of prime numbers.
27. Let \( n \in \mathbb{N} \) such that \( 2^n + 1 \) is prime. Prove that \( n \) is prime.

We prove the contrapositive holds. Suppose that \( n \) is not prime, and let \( a, b \in \mathbb{N} \) such that \( n = ab \) and \( a, b > 1 \). Then:

\[
2^n + 1 = (2^a)^b + 1
= (2^a - 1)(2^{a(b-1)} + 2^{a(b-2)} + \ldots + 2^a + 1)
\]

so \( 2^n + 1 \) is not prime.

Bonus: Is the converse true?

No, because 3 is prime but \( 2^3 + 1 = 9 \) which is not prime.

Note that primes of the form \( 2^n + 1 \) are called Mersenne primes.

Probability

3. There are 20 red, 15 green, and 40 yellow balls in a bin. Ten balls are selected uniformly at random.

(a) What is the probability that exactly 5 red balls will be selected?

(b) What is the probability that exactly 5 red balls will be selected given that exactly 4 balls are yellow.
(a) \( P(\text{exactly 5 red}) = \frac{\binom{20}{5} \binom{55}{5}}{\binom{75}{10}} \)

(b) \( P(\text{exactly 5 red} | \text{exactly 4 yellow}) \)

\[
P(\text{exactly 5 red} \& \text{exactly 4 yellow}) = \frac{P(\text{exactly 4 yellow})}{P(\text{exactly 4 yellow})}
\]

\[
= \frac{\frac{\binom{20}{5} \binom{50}{4} \binom{15}{1}}{\binom{75}{10}}}{\frac{\binom{40}{4} \binom{35}{6}}{\binom{75}{10}}}
\]

\[
= \frac{15 \cdot \frac{\binom{20}{5}}{\binom{55}{6}}}{1}
\]

There are three closed boxes on the table:

→ The first box has 2 red and 3 black balls.
→ The second box has 3 red and 2 black balls.
→ The third box has 5 red balls.

Rolls a die.

→ If the result is 1 then they open the first box.
→ If the result is even then they open the second box.
→ If the result is 3 or 5 then they open the third box.

After that they pick a ball from the open box.
What is the probability that the other four balls in the open box are red given that the selected ball is red?

\[ P(\text{third box} \mid \text{red}) = \]

Bayes' formula:

\[ P(\text{red} \mid \text{third box}) P(\text{third box}) \]

\[ = \frac{P(\text{red} \mid \text{first box}) P(\text{first box}) + P(\text{red} \mid \text{second box}) P(\text{second box}) + P(\text{red} \mid \text{third box}) P(\text{third box})}{1 \cdot \frac{2}{6}} \]

\[ = \frac{\left( \frac{2}{5} \cdot \frac{1}{6} + \frac{3}{5} \cdot \frac{1}{2} + 1 \cdot \frac{2}{6} \right) \cdot \frac{10}{30}}{\frac{2}{30} + \frac{9}{30} + \frac{10}{30}} \]

\[ = \frac{10}{21} \]