Divisibility

Let \( n \in \mathbb{N} \) with \( n > 1 \) and let \( n = \prod_{i=1}^{k} p_i^{l_i} \) be the prime factorization of \( n \). Prove that \( n \) has exactly \( (l_1 + 1)(l_2 + 1) \cdots (l_k + 1) \) distinct divisors.

Proof: First we show that every divisor of \( n \) has the form \( \prod_{i=1}^{k} p_i^{r_i} \) for some \( 0 \leq r_i \leq l_i \).

Clearly every number of that form is a divisor of \( n \), so we only need to show that the converse holds. So let \( d \) be a divisor of \( n \). We want to show that \( d \) is of the form above. This follows from the transitivity of divisibility:

* if \( p_i \) divides \( d \) for some \( p_i \not\in \{p_1, \ldots, p_k\} \) then, since \( d \) divides \( n \), \( p_i \) must divide \( n \), a contradiction.

* if \( q_i \) divides \( d \) for some \( q_i > l_i \) then \( q_i \) must divide \( n \), a contradiction.

Either way, we have a contradiction so \( d \) must be
of the form \( p_1^{r_1} \cdots p_k^{r_k} \) for some \( 0 \leq r_i < l_i \).

To enumerate the divisors of \( m \) is thus equivalent to choosing, for each \( i \), a number \( r_i \) amongst the \( l_i + 1 \) numbers \( \{0, 1, \ldots, l_i + 1\} \). By the rule of product there are thus \( (l_1 + 1) \cdots (l_k + 1) \) distinct divisors of \( m \).

**Examples:**

\[ \times 62 = 2 \times 31 \] has \( 2 \times 2 = 4 \) distinct divisors: \( 1, 2, 31, \text{and } 62 \).

\[ \times 24 = 2^3 \times 3 \] has \( 4 \times 2 = 8 \) distinct divisors: \( 1, 2, 4, 8, 3, 6, 12, \text{and } 24 \).

\[ \times 36 = 2^2 \times 3^2 \] has \( 3 \times 3 = 9 \) distinct divisors: \( 1, 2, 4, 3, 6, 12, 9, 18, \text{and } 36 \).
2. Construct two different numbers with exactly 12 divisors.

Here are some numbers with exactly 12 divisors:

* $400 = 2^3 \times 5^2$ (because $4 \times 2 = 12$, soley problem 1 above, it has exactly 12 distinct divisors)
* $108 = 2^2 \times 3^3$ (because $3 \times 4 = 12$)
* $96 = 2^5 \times 3$ (because $6 \times 2 = 12$)

3. Construct:

(a) a number with exactly 12 distinct divisors and exactly two prime factors.

(b) a number with exactly 12 distinct divisors and exactly three prime factors.

(a) $96 = 2^5 \times 3$ has exactly 12 distinct divisors (because $6 \times 2 = 12$) and exactly two prime factors.

(b) $60 = 2^2 \times 3 \times 5$ has exactly 12 distinct divisors (because $3 \times 2 \times 2 = 12$) and exactly three prime factors.
Let \( m \in \mathbb{N} \) and let \( p \) be a prime number. Prove that

\[
\sum_{i=1}^{\lfloor \frac{m}{p} \rfloor} \left\lfloor \frac{m}{p^i} \right\rfloor. \quad \text{(This is called Legendre's formula)}
\]

Observe that:

\[
\text{exponent of } p \text{ in the prime factorization of } n! = \sum_{j=1}^{\lfloor \frac{m}{p} \rfloor} \text{exponent of } p \text{ in the prime factorization of } j
\]

\[
= \sum_{j=1}^{\lfloor \frac{m}{p} \rfloor} \sum_{i=1}^{\text{exponent of } p \text{ in the prime factorization of } j} (p, j)
\]

where we defined

\[
(p, j) = \begin{cases} 
1 & \text{if } p^i \mid j \\
0 & \text{otherwise}
\end{cases}
\]

Therefore, swapping the two sums we see that:

\[
\text{exponent of } p \text{ in the prime factorization of } n! = \sum_{i=1}^{\lfloor \frac{m}{p} \rfloor} \sum_{j=1}^{\frac{m}{p^i}} (p, j)
\]

\[
= \frac{m}{p^i}
\]

where \( p^i \) divides \( j \).

Indeed, the exponent of \( p \) in the factorization of \( n \) is 
\[ \sum_{d|n} \left\lfloor \frac{n}{d} \right\rfloor. \]

Example: \( n = 9 \), \( p = 2 \). The table below shows that the exponent of \( 2 \) in the factorization of \( n \) is 7.

<table>
<thead>
<tr>
<th>Multiple of 2</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>How many?</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>X</td>
<td></td>
<td>![X]</td>
<td></td>
<td>![X]</td>
<td></td>
<td>![X]</td>
<td>![X]</td>
<td>( \lfloor 9/2 \rfloor = 4 )</td>
</tr>
<tr>
<td>Multiple of 4</td>
<td>![X]</td>
<td>![X]</td>
<td>![X]</td>
<td>![X]</td>
<td>![X]</td>
<td>![X]</td>
<td>![X]</td>
<td>![X]</td>
<td>![X]</td>
<td>( \lfloor 9/4 \rfloor = 2 )</td>
</tr>
<tr>
<td>Multiple of 8</td>
<td>![X]</td>
<td>![X]</td>
<td>![X]</td>
<td>![X]</td>
<td>![X]</td>
<td>![X]</td>
<td>![X]</td>
<td>![X]</td>
<td>![X]</td>
<td>( \lfloor 9/8 \rfloor = 1 )</td>
</tr>
</tbody>
</table>

In * above, we sum along the columns of the table, then along its rows while in ** we sum along the rows of the table first, then along its columns.

Find the largest integer \( k \) such that \( 5^k \) divides \( 250! \).

This corresponds to the case \( n = 250 \) and \( p = 5 \) in Legendre's formula.

Observe that:
\[ \left\lfloor \frac{250}{5} \right\rfloor = 50 \]
\[ \left\lfloor \frac{250}{25} \right\rfloor = 10 \]
\[ \left\lfloor \frac{250}{125} \right\rfloor = 2 \]
\[ \left\lfloor \frac{250}{625} \right\rfloor = 0 \text{ if } d > 4 \]
The largest integer \( b \) such that \( 5^b \) divides \( 250! \) is therefore

\[ 50 + 10 + 2 = 62. \]