The Principle of Inclusion - Exclusion

Recall: \[ \left| \bigcup_{i=1}^{n} S_i \right| = \sum_{j=1}^{n} (-1)^{j+1} \sum_{i_1 < i_2 < \ldots < i_j} \left| S_{i_1} \cap S_{i_2} \cap \ldots \cap S_{i_j} \right| \]

This is an additional wrinkle on a problem done before. Three couples, the Smiths, Joneses, and Murphys, are going to form a line.

(a) In how many such lines will Mr. and Mrs. Jones be next to each other?

\[ 2 \times 5! \]

(b) In how many such lines will Mr. and Mrs. Jones be next to each other and Mr. and Mrs. Murphy be next to each other?

\[ 2^2 \times 4! \]

(c) In how many such lines will all three couples be next to each other?

\[ 2^3 \times 3! \]
(2) In how many such lines will at least one couple lie next to each other?

Let \( S = \{ \text{lines where Mr. and Mrs. Smith are adjacent} \} \),

\( J = \{ \text{---//--- Jones \quad ---//---} \} \),

and \( M = \{ \text{---//--- Murphy \quad ---//---} \} \).

By the principle of inclusion - exclusion:

\[
|S \cup J \cup M| = |S| + |J| + |M| - (|S \cap J| + |S \cap M| + |J \cap M|) + |S \cap J \cap M|
\]

\[
= 3 \times 2 \times 5! - 3 \times 2^6 \times 4! + 2^3 \times 3!
\]

\[
= (6 \times 5 - 3 \times 4 + 2) \times 4!
\]

\[
= 20 \times 4!
\]

[2] Given the five types of coins (pennies, nickels, dimes, quarters, and half-dollars), in how many ways can one select \( n \) coins so that no coin is selected more than 4 times?
Note that the negation of
"for every type of coin, there are at most 4 coins of that type"
is "there exists a type of coin such that there are
at least 5 coins of that type."

Therefore, if we define
\[ S_i = \{ \text{collections of } n \text{ coins with at least} \]
\[ \text{five coins of type } i \} \]

then we seek to enumerate the set
\[ (S_1, oS_2, oS_3, oS_4, oS_5) = U \cup (S_1, oS_2, oS_3, oS_4, oS_5) \]

where
\[ U = \{ \text{collections of } n \text{ coins} \} \]

By the principle of inclusion-exclusion:
\[ |U \cup (S_1, oS_2, oS_3, oS_4, oS_5)| = |U| - \sum_{i} |S_i| + \sum_{i < j} |S_i \cap S_j| \]
\[ - \sum_{i < j < k} |S_i \cap S_j \cap S_k| + \sum_{i < j < k < \ell} |S_i \cap S_j \cap S_k \cap S_\ell| \]
\[ - |S_1 \cap S_2 \cap S_3 \cap S_4 \cap S_5| \]

where, by "stars & bars" or "piblads & golds," or by identical
marbles into distinct boxes:
\[ |U| = \binom{n+5-1}{5-1} = \binom{n+4}{4} \]
\[ 1S_i = \binom{n-5+(5-1)}{5-1} = \binom{n-1}{4} \]
\[ 1S_i \circ 1S_j = \binom{(n-2\cdot 5)+(5-1)}{5-1} = \binom{n-6}{4} \]
\[ 1S_i \circ 1S_j \circ 1S_k = \binom{(n-3\cdot 5)+(5-1)}{5-1} = \binom{n-11}{4} \]
\[ 1S_i \circ 1S_j \circ 1S_k \circ 1S_l = \binom{(n-4\cdot 5)+(5-1)}{5-1} = \binom{n-16}{4} \]
\[ 1S_i \circ 1S_j \circ 1S_k \circ 1S_l \circ 1S_m = \binom{(n-5\cdot 5)+(5-1)}{5-1} = \binom{n-21}{4} \]

So finally:
\[ |U \setminus (S_1 \cup \ldots \cup S_5)| = \binom{n+4}{4} - 5 \binom{n-1}{4} + 2 \binom{5}{2} \binom{n-6}{4} \]
\[ - \binom{5}{3} \binom{n-11}{4} + \binom{5}{4} \binom{n-16}{4} - \binom{n-21}{4} \]

\[ \boxed{3 \text{ count the number of surjections from } [m] \text{ to } [m].} \]

Note that
\[ \{ \text{surjective functions} \} \setminus \{ \text{all functions} \} = \{ \text{functions which are not surjective} \}. \]
Crucially:

\[ |\{ f: [m] \to [m] \text{ such that } f \text{ is not surjective} \}| = \bigcup_{\sigma \in [m]} |\{ f: [m] \to [m] \text{ such that } 0 \notin \text{Im} f \}| \]

...and we may use the principle of inclusion-exclusion to count the numbers of elements in that set.

Indeed:

\[ |\{ f: [m] \to [m] \text{ such that } f \text{ is not surjective} \}| = \bigcup_{\sigma \in [m]} |\{ f: [m] \to [m] \text{ such that } 0 \notin \text{Im} f \}| \]

\[ = \sum_{\sigma \in [m]} |\sigma_0| - \sum_{\{\sigma_0, \sigma_1 \in [m]\}} |\sigma_0 \circ \sigma_1| + ... \]

\[ = \sum_{j=1}^{\infty} (-1)^{j+1} \sum_{\{\sigma_1, ..., \sigma_j \in [m]\}} |\sigma_0 \circ ... \circ \sigma_j| \]

\[ = \sum_{j=1}^{\infty} (-1)^{j+1} \sum_{\{\sigma_1, ..., \sigma_j \in [m]\}} |\sigma_0 \circ ... \circ \sigma_j| \]

\[ = |\{ \text{functions from } [m] \text{ to } [m] \text{ whose image does not contain } j \text{ elements of } [m]| \]

\[ = (\binom{m}{j})(m-j)^m \]
numbers of functions from $[m]$ to a set with $(m-j)$ elements

\[
\binom{m}{j} \binom{m-j}{m-j}^n
\]

number of ways
to choose the $j$

missing elements

So finally, since $|\{\text{all functions from } [m] \to [m]\}| = m^n$,
which can be written as $(-1)^0 \binom{m}{0} (m-0)^n$, we have:

\[
|\{\text{surjective functions from } [m] \to [m]\}| = m^n - \sum_{j=1}^{\infty} (-1)^{j+1} \binom{m}{j} (m-j)^n
\]

\[
= (-1)^0 \binom{m}{0} (m-0)^n + \sum_{j=1}^{\infty} (-1)^j \binom{m}{j} (m-j)^n
\]

\[
= \sum_{j=0}^{\infty} (-1)^j \binom{m}{j} (m-j)^n.
\]