Theorem (Bernstein - Cantor - Schröder)

Let A and B be sets. If there exist injections \( f: A \rightarrow B \) and \( g: B \rightarrow A \) then there exists a bijection \( h: A \rightarrow B \).

Proof

(a) Preliminary definitions

We call an element \( b \) of \( B \) lonely if there is no element \( a \in A \) such that \( f(a) = b \). We say that an element \( b_0 \) of \( B \) is a descendant of an element \( b_0 \) of \( B \) if there is a natural number \( n \geq 0 \) such that \( b_0 = (f \circ g)^n(b_0) \).

(*) Note that if an element is lonely then it is a descendant of a lonely element (itself).

(b) Defining the function \( h \)

Define \( h: A \rightarrow B \) as follows:

\[
h(a) = \begin{cases} 
  g^{-1}(a) & \text{if } f(a) \text{ is the descendant of a lonely point} \\
  f(a) & \text{otherwise} 
\end{cases}
\]
Note that if \( f(a) \) is the descendent of a lonely point then \( f(a) = f(g(b)) \) for some \( b \in B \). Since \( f \) is injective it follows that \( a = g(b) \), so \( a \) is in the image of \( g \). Since \( g \) is injective, \( g^{-1}(a) \) is therefore well-defined. Thus the function \( h \) is well-defined.

**(2) Surjectivity of \( h \)**

Let \( b \in B \). We break into cases.

**Case 1:** \( b \) is the descendent of a lonely point.

Then \( (f \circ g)(b) \) is also the descendent of a lonely point, so \( h(g(b)) = g^{-1}(g(b)) = b \).

\[ h(g(b)) = b \]

By definition of \( h \).

**Case 2:** \( b \) is not the descendent of a lonely point.

Then, by (\#) above, \( b \) is not lonely, and so by definition of lonely points there must be some \( a \in A \) such that \( g(a) = b \). Then, by definition of \( h \), \( h(a) = b \).

We have thus shown that \( h \) is surjective.
(**) Injectivity of $h$

Note that, for any $a \in A$, $h(a)$ is a descendent of a lonely point if and only if $f(a)$ is a descendent of a lonely point.

Why? Suppose $h(a)$ is a descendent of a lonely point and suppose, for the sake of contradiction, that $f(a)$ is not a descendent of a lonely point. Then, by definition of $h$, $h(a) = f(a)$, which is a contradiction. Conversely, suppose that $f(a)$ is a descendent of a lonely point. Then $f(a) = f(g(b))$, where $b$ is the descendent of a lonely point. Therefore, by definition of $h$, $h(a) = g^{-1}(a) = g^{-1}(g(b)) = b$, because $f$ is injective.

So indeed, $f$ is a descendent of a lonely point.

So now suppose that we have two elements $a_1, a_2 \in A$ such that $h(a_1) = h(a_2)$. We break into cases.

(***)
Case 1: \( f(a_1) \) is the descendant of a lonely point.

Then, by \((**)\) and \((***)\), so is \( f(a_2) \). Therefore
\[
g^{-1}(a_1) = h(a_1) = h(a_2) = g^{-1}(a_2).
\]
Since \( g \) is a well-defined function, it follows that \( a_1 = a_2 \)
(i.e. \( a_1 = g(g^{-1}(a_1)) = g(g^{-1}(a_2)) = a_2 \)).

Case 2: \( f(a_1) \) is not the descendant of a lonely point.

Then, by \((**)\) and \((***)\), neither is \( f(a_2) \). Therefore
\[
f(a_1) = h(a_1) = h(a_2) = f(a_2).
\]
Since \( f \) is injective, we deduce that \( a_1 = a_2 \).

Either way: \( a_1 = a_2 \). So indeed \( h \) is injective.

Since \( h \) is both injective and surjective, it is bijective, as desired.
A finite binary string is a finite sequence of 0's and 1's.

(a) How many binary strings of length \( n \) are there?

We can enumerate all binary strings of length \( n \) by repeating the same step, which is to choose either 0 or 1 \( n \) times. By the Rule of Product, there are therefore \( 2^n \) such strings.

(b) Let \( m, k \in \mathbb{N} \) with \( k < m \). How many binary strings of length \( m \) have exactly \( k \) ones?

There are \( \binom{m}{k} \) such binary strings because we can put binary strings of length \( m \) with exactly \( k \) ones in a one-to-one correspondence with subsets of size \( k \) of \( \{1, \ldots, m\} \), and there are \( \binom{m}{k} \) such subsets.

Indeed, let \( \{i_1, \ldots, i_k\} \) be a subset of \( \{1, \ldots, m\} \) of size \( k \). To this subset we associate the string \( s \) whose \( j \)-th element \( s_j \) is given by:

\[
    s_j = \begin{cases} 
        1 & \text{if } j \in \{i_1, \ldots, i_k\} \\
        0 & \text{otherwise}
    \end{cases}
\]

and note that the map \( \{i_1, \ldots, i_k\} \rightarrow s \) is bijective.
(c) How many ternary strings (finite sequences of 0's, 1's, and 2's)
of length \( n \) are there?

We can enumerate all ternary strings of length \( n \) by repeating
the same step, which is to choose either 0, 1, or 2, \( n \) times.
By the Rule of Product, there are therefore \( 3^n \) such strings.

(d) Let \( n, b \in \mathbb{N} \) with \( b < n \). How many ternary strings of
length \( n \) have exactly \( b \) ones?

We proceed in two steps:

Step 1: Choose which entries of the string will be ones.

As per part (c) above, there are \( \binom{n}{b} \) ways to
do this.

Step 2: Assign the values 0 or 2 to each of the \( n-b \)
remaining entries of the string. By the Rule of Product
there are \( 2^{n-b} \) ways to do this.

Therefore, by the Rule of Product there are \( \binom{n}{b} 2^{n-b} \) ternary
strings of length \( n \) with exactly \( b \) ones.
(a) A man has 10 distinct candies and he puts them in two distinct bags such that each bag contains 5 candies. In how many ways can he do it?

Since the content of the second bag is determined by the content of the first bag, it is enough to count the number of ways he has to fill the first bag. Since the candies are distinct, this amounts to counting subsets of size 5 of the set \{candy 1, candy 2, ..., candy 10\}. There are \(\binom{10}{5}\) such subsets, and thus \(\binom{10}{5}\) ways to do it.

(b) A man has 10 identical candies and he puts them in two distinct bags such that each bag contains 5 candies. In how many ways can he do it?

Each of bag 1 and bag 2 will contain 5 identical (i.e. indistinguishable) candies, so there is only one way to do this.
(a) A woman has 10 distinct candies and she puts them in two identical bags such that each bag has 5 candies. In how many ways can she do it?

As shown in (a), there are \( \binom{10}{5} \) ways to do this if the bags are distinct. This counts every possibility twice since the two bags could be labelled as (Bag 1, Bag 2) or (Bag 2, Bag 1).

There are therefore \( \frac{\binom{10}{5}}{2} \) ways to do this.

(b) A woman has 10 identical candies and she puts them in two identical bags such that no bag is empty. In how many ways can she do it?

There are 5 ways to write 10 as a sum of two integers greater than or equal to one, and therefore 5 ways to do this.

\[
10 = 1 + 9 \\
= 2 + 8 \\
= 3 + 7 \\
= 4 + 6 \\
= 5 + 5.
\]

Note: There is no general formula to count the number of ways \( m \cdot n \) can be written as the sum of \( k \) integers \( \geq 1 \).