Show that \( f \) from \([2, \infty)\) to \([-3, \infty)\) defined by 
\[
f(x) = x^2 - 4x + 1
\]
is a bijection.

**Proof #1:** We prove that \( f \) is surjective and injective.

**Surjectivity:** Let \( y \in [-3, \infty) \) and define \( x = 2 + \sqrt{y+3} \). Note that \( \sqrt{y+3} \) is well-defined since \( y \geq -3 \) and that \( x \geq 2 \).

Now since, by completing the square,
\[
f(x) = x^2 - 4x + 1 = (x-2)^2 - 3,
\]
we deduce that \( f(x) = y \). So indeed \( f \) is surjective.

**Injectivity:** Let \( x_1, x_2 \in [2, \infty) \) such that \( f(x_1) = f(x_2) \).

By completing the square as above we see that:
\[
f(x_1) = f(x_2) \iff (x_1-2)^2 + 3 = (x_2-2)^2 + 3
\]

\[
\iff (x_1-2)^2 = (x_2-2)^2
\]

\[
\iff \sqrt{(x_1-2)^2} = \sqrt{(x_2-2)^2}
\]

\[
\iff |x_1-2| = |x_2-2|
\]

By the assumption \( x_1, x_2 \geq 2 \) we have:

\[
|x_1-2| = x_1 - 2 \quad \text{and} \quad |x_2-2| = x_2 - 2
\]

Thus,
\[
x_1 - 2 = x_2 - 2 \quad \text{since} \quad x_1, x_2 \geq 2
\]

\[
\Rightarrow x_1 = x_2
\]
So indeed, \( f \) is an injection.

**Proof #2:** We prove that \( f \) is invertible.

Define \( g: [2, \infty) \to [2, \infty) \) by \( g(y) = 2 + \sqrt{y + 3} \).

Note that \( g \) is a well-defined function since \( y \geq -3 \) and hence \( \sqrt{y + 3} \) is well-defined, and since \( g(y) \geq 2 \) for every \( y \geq -3 \). By completing the square we see that \( f(x) = x^2 - 4x + 1 = (x - 2)^2 - 3 \).

Therefore: for every \( x \geq 2 \),

\[
g(f(x)) = 2 + \sqrt{(x - 2)^2 - 3} = 2 + |x - 2| = 2 + (x - 2) = x
\]

since \( x \geq 2 \).

and, for every \( y \geq -3 \),

\[
f(g(y)) = (\sqrt{y + 3})^2 - 3 = y.
\]

This shows that \( g \) is an inverse of \( f \), and hence \( f \) is invertible.
2. Show that \( f \) from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \) defined by
\[
f(x, y) = (x + y, x + y)
\]
is not a bijection.

**Proof \#1:** \( f \) is not injective because \( f(1, 0) = f(0, 1) = (1, 1) \).

**Proof \#2:** \( f \) is not surjective because \((0, 1) \notin \text{Im}(f)\).
Indeed, suppose for the sake of contradiction that \((0, 1) \in \text{Im}(f)\). Then \((0, 1) = f(x, y)\) for some \((x, y) \in \mathbb{R}^2\)
and so, in particular, \(0 = x + y = 1\), which is a contradiction.
Therefore \((0, 1) \notin \text{Im}(f)\).

3. Show that \( f \) from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \) defined by
\[
f(x, y) = (x + y, x - y)
\]
is a bijection.

Consider the function \( g : \mathbb{R}^2 \to \mathbb{R}^2 \) defined by
\[
g(a, b) = \left( \frac{a + b}{2}, \frac{a - b}{2} \right).
\]
Then, for every \((x, y) \in \mathbb{R}^2\),
\[
g \circ f(x, y) = g(x + y, x - y) = \left( \frac{(x + y) + (x - y)}{2}, \frac{(x + y) - (x - y)}{2} \right) = (x, y)
\]
and, for every \((a, b) \in \mathbb{R}^2\),
\[(f \circ g)(a, b) = f \left( \frac{a+b}{2}, \frac{a-b}{2} \right) = \left( \frac{a+b}{2}, \frac{a-b}{2} \right) = (a, b) \]

i.e. \( g \) is an inverse of \( f \). We may conclude that \( f \) is invertible, and hence bijective.

4. Let \( A \) be the set of all subsets of \([n, 1]\) with an even number of elements, and let \( B \) be the set of all subsets of \([n, 1]\) with an odd number of elements.

Find a bijection from \( A \) to \( B \).

Define \( f: A \rightarrow B \) by

\[
 f(S) = \begin{cases} 
 S \setminus \{0\} & \text{if } 0 \in S \\
 S \setminus \{0\} & \text{if } 0 \notin S
\end{cases}
\]

Note that since the size of \( f(S) \) differs from the size of \( S \) by \(+1\) or \(-1\), \( f \) is well-defined.

Now let us prove that \( f \) is both injective and surjective.

Injectivity: Let \( S, T \in A \) such that \( f(S) = f(T) \).

\( \text{(case 1: } 0 \in f(S) \text{.} \) 

Then, by definition of \( f \), \( 0 \notin S \). Since \( f(S) = f(T) \)
it follows that \( 0 \notin f(1) \), and hence \( O \notin S \). In particular, we now know that \( f(S) = f(T) \) can be written as \( S \cup \{0\} = T \cup \{0\} \). (*)

Now let us deduce from (*) that \( S = T \). First we prove that \( S \subseteq T \). Let \( x \in S \). Since \( S \subseteq S \cup \{0\} \), we deduce from (*) that \( x \in T \cup \{0\} \). Since \( x \in S \) and \( S \) does not contain \( 0 \), it follows that \( x \in T \).

So indeed \( S \subseteq T \). Using (*) once more we can show in exactly the same way that \( T \subseteq S \), hence \( S = T \).

Case 2: \( 0 \notin f(S) \).

Then, once again by definition of \( f \), \( O \subseteq S \). Since \( f(S) = f(T) \) it follows that \( 0 \notin f(T) \), and hence \( O \subseteq T \). We thus know that \( f(S) = f(T) \) can be written as \( S \cup \{0\} = T \cup \{0\} \). (**) 

Now let us deduce from (**) that \( S = T \). This is immediate since \( S = (S \setminus \{0\}) \cup \{0\} = (T \setminus \{0\}) \cup \{0\} = T \).

Since \( O \subseteq S \) and \( O \subseteq T \) since \( O \subseteq T \)

So indeed \( f \) is injective.
Surjectivity: Let $S \in B$.

Case 1: $0 \in S$. Then $S \cdot 101 \in A$ and $f(S \cdot 101) = S$.

Case 2: $0 \notin S$. Then $S \cdot 01 \in A$ and $f(S \cdot 01) = S$.

Either way we may find a set $T \in A$ such that $f(T) = S$, so indeed $f$ is surjective.

Alternative proof that $f$ is a bijection:

Define $g : B \to A$ by $g(S) = \begin{cases} S \cdot 01 & \text{if } 0 \in S \\ S \cdot 101 & \text{if } 0 \notin S \end{cases}$

Then $g$ is an inverse of $f$, hence $f$ is invertible and thus bijective.