More induction

(1) A "toonie" is a Canadian 2$ coin. Consider the following proof that, for any integer \( n \geq 2 \), \( n \) Canadian dollars can be paid in toonies.

Base case: \( n = 2 \) can be paid with one toonie.

Induction hypothesis: suppose that for some integer \( h \geq 2 \), \( h \) Canadian dollars can be paid in toonies for any \( 2 \leq i \leq h \).

Then, since \( h+1 = (h-1) + 2 \) and since, by the induction hypothesis, \( h-1 \) Canadian dollars can be paid for in toonies, we may add one toonie and pay \( h+1 \) Canadian dollars in toonies.

What is wrong with this proof?
The issue is that, if we call \( P(k) \) the statement "\( k \) Canadian dollars can be paid in toonies", then we have shown in the induction step that \( P(k-1) \Rightarrow P(k+1) \). Since we have shown, in the base case, that \( P(2) \) is true, we may deduce that \( P(k) \) holds for any even integer \( k \geq 2 \), but we know nothing about \( P(b) \) when \( b \) is odd!

2) Consider a sequence of integers \( a_n, n \in \mathbb{N} \), where \( a_1 \) and \( a_2 \) are odd and where
\[
a_n = 2a_{n-1} + 3a_{n-2} \quad \text{for any } n \geq 3.
\]
(a) Prove that \( a_n \) is odd for all \( n \in \mathbb{N} \).

We proceed by induction.

Base cases: \( n = 1, 2 \)

We know that \( a_1 \) and \( a_2 \) are odd by assumption.
Induction step:

Let $k$ be some integer with $k \geq 2$ and suppose that $a_i$ is odd for every integer $i=1,\ldots,k$.

In particular, $a_2$ and $a_{k-1}$ are odd so there exist $x, y \in \mathbb{Z}$ such that $a_2 = 2x + 1$ and $a_{k-1} = 2y + 1$. Therefore:

\[
\begin{align*}
a_{k+1} &= 2a_2 + 3a_{k-1} \\
&= 2(2x+1) + 3(2y+1) \\
&= 4x + 6y + 5 \\
&= 2(2x+3y+2) + 1
\end{align*}
\]

i.e. indeed $a_{k+1}$ is odd.

We have thus proven by induction that $a_n$ is odd for every $n \in \mathbb{N}$.

(b) Suppose now that $a_1 = a_2 = 1$. Prove that

\[a_n = \frac{1}{2} (3^{n-1} - (-1)^n)\]

for every $n \in \mathbb{N}$.

We proceed by induction.
Base cases: \( m = 1, 2 \)

Observe that indeed
\[
\begin{align*}
a_1 &= \frac{1}{2} (1+1) = \frac{1}{2} \left( 3^0 - (-1)^1 \right) \\
a_2 &= \frac{1}{2} (3-1) = \frac{1}{2} \left( 3^1 - (-1)^2 \right)
\end{align*}
\]

Induction step:
Let \( b \in \mathbb{N} \) be some integer with \( b \geq 2 \) and suppose that \( a_x = \frac{1}{2} \left( 3^{x-1} - (-1)^x \right) \) for every \( x = 1, \ldots, b \).

Then \( a_{b+1} = 2a_b + 3a_{b-1} \)
\[
= 2 \cdot \frac{1}{2} \left( 3^{b-1} - (-1)^b \right) + 3 \cdot \frac{1}{2} \left( 3^{b-2} - (-1)^{b-1} \right)
\]
by the induction hypothesis
\[
= \frac{3}{2} 3^{b-1} + \frac{1}{2} \cdot (-1)^b + \frac{3}{2} (-1)^{b-1}
\]
\[
= \frac{1}{2} \left( 3^b - (-1)^{b+1} \right)
\]
i.e. indeed \( a_{b+1} = \frac{1}{2} \left( 3^{b+1} - (-1)^{b+1} \right) \)

We have thus proven by induction that \( a_m = \frac{1}{2} \left( 3^{m-1} - (-1)^{m} \right) \)
for every \( m \in \mathbb{N} \).
(3) Prove that every integer amount of \( n \geq 18 \) dollars can be paid by using 4 or 7 dollar bills only.

We proceed by induction. Let \( P(k) \) denote the statement "\( k \) dollars can be paid using 4 or 7 dollar bills only".

**Induction step:** We prove that, for any integer \( k > 0 \), \( P(k) \Rightarrow P(k + 4) \).

Suppose that we can pay \( k \) dollars using 4 or 7 dollar bills only, for some \( k > 0 \). By adding a 4 dollar bill we may also pay \( k + 4 \) dollars, i.e. indeed \( P(k) \Rightarrow P(k + 4) \).

**Base cases:** Because we have proven in our induction step that \( P(k) \Rightarrow P(k + 4) \) we need 4 base cases.

So consider \( n = 18, 19, 20, 21 \):
18 = 4 + 2 \cdot 7  \\
19 = 3 \cdot 4 + 7  \\
20 = 5 \cdot 4  \\
21 = 3 \cdot 7  \\

We have thus proven by induction that \( P(n) \) holds for any \( n > 18 \), i.e., \( n \) dollars can be paid using 4 or 7 dollar bills only for any \( n \geq 18 \).

(4) Consider \( n \) married couples at a party. Suppose that no person shakes hands with their spouse, and the \( 2n-1 \) people other than the host shake hands with different numbers of people. With how many people does the host shake hands?

The host shakes hands with \( n-1 \) people.

We prove this claim by induction.
Base case: \( n = 1 \)

If there is one couple, then since the host cannot shake hands with their spouse it follows that they have shaken no hands.

**Induction step:** Suppose that if \( h \) couples are in the situation described above then the host shook \( h + 1 \) hands, for some integer \( h \geq 1 \).

Consider \( h + 1 \) couples in the situation described above.

Note that for every individual, there are two people that they cannot shake hands with (themselves & their spouse). When asked how many hands they have shaken, everyone will thus give an answer between 0 and \( 2h \).

\( \text{(since} \quad 2h = (2h + 2) - 2 \text{)} \)

number of people at the party
Since the $2^k + 1$ people other than the host give different answers, and since there are $2^k + 1$ possible answers, we know that for each integer $i$ with $0 \leq i \leq 2^k$, exactly one person has shaken hands with $i$ people. We will call this person "Person $i$".

The key point is that Person $2^k$ and Person $0$ must form a couple (because Person $2^k$ shook hands with everyone except their spouse and themselves) and moreover if we remove this couple from the party, then we are left with $2^k$ couples in the same situation as that described above.

Therefore, by the induction hypothesis, the host shook $2 - 1$ hands with people from these $2^k$ couples. Since the host also shook the hand of Person $2^k$, but not of Person $0$, it follows that they shook $(2^k + 1) - 2 = 2^k$ hands.