Instability of an anisotropic micropolar fluid

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Spot the difference resemblance

(a) Blood
(b) Sperm
(c) Liquid crystal

(a) Ferrofluid
(b) Knee joint
(c) Polymer melt

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Model and setup

The PDE

Result

Difficulties and strategy
Micropolar fluids and continuum mechanics

Full kinematic description: flow map $\eta$, initial micro-inertia ($J_0$), and micro-rotation map $Q$.

Unknowns: velocity $u$, micro-inertia $J$, and angular velocity $\omega$. 
Micropolar fluids and continuum mechanics

How to derive the Navier-Stokes equations

1. Conservation laws
   - Mass
   - Linear momentum
   - Angular momentum

2. Linear stress-strain relation
   - Stress $T$ linear in $\mathbb{D}u$

3. Frame-invariance

$$(1) - (3) \Rightarrow \text{(incompressible) Navier-Stokes}$$
Micropolar fluids and continuum mechanics

How to derive the equations of micropolar fluids

1. Conservation laws
   - Mass
   - Linear momentum
   - Angular momentum
   - Micro-inertia

2. Linear stress-strain relation
   - Stress $T$ linear in $(\mathbb{D}u, \frac{1}{2} \nabla \times u - \omega)$
   - Couple-stress $M$ linear in $\nabla \omega$

3. Frame-invariance

   $(1) - (3) \Rightarrow \text{incompressible micropolar fluids}$
Our setup

- The microstructure is **anisotropic** but has an **axis of symmetry**.

  - Isotropic
  - Rod-like
  - Pancake-like
  - ‘Fully anisotropic’

- There is a **constant micro-torque** acting on the microstructure.
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The equations are

\[
\begin{align*}
\partial_t u + (u \cdot \nabla) u &= \mu \Delta u + \kappa \nabla \times \omega - \nabla p, & \text{on } \mathbb{T}^3 \\
\nabla \cdot u &= 0, & \text{on } \mathbb{T}^3 \\
J (\partial_t \omega + (u \cdot \nabla) \omega) + \omega \times J \omega &= \tau e_3 + \kappa \nabla \times u - 2\kappa \omega + (\alpha - \gamma) \nabla (\nabla \cdot \omega) + \gamma \Delta \omega & \text{on } \mathbb{T}^3 \\
\partial_t J + (u \cdot \nabla) J &= [\Omega, J] & \text{on } \mathbb{T}^3
\end{align*}
\]

where \( \omega = \text{vec } \Omega \), i.e. \( \Omega \mathbf{v} = \omega \times \mathbf{v} \) for any \( \mathbf{v} \in \mathbb{R}^3 \). \( \alpha, \gamma, \kappa, \mu > 0 \) are viscosity constants and \( \tau > 0 \) is the magnitude of the external micro-torque.

This is supplemented by initial conditions \((u_0, p_0, \omega_0, J_0)\).
Model and setup

The PDE

Result

Difficulties and strategy
Result

In the presence of a constant micro-torque and provided the microstructure has an axis of symmetry the system has a unique equilibrium

Theorem

- If the microstructure is rod-like then the equilibrium is non-linearly unstable in $L^2$.
- If the microstructure is pancake-like then the equilibrium is non-linearly stable in $H^s$ with algebraic decay to equilibrium.
Model and setup

The PDE

Result

Difficulties and strategy

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The role of anisotropy – Unstable case

Strategy

- Find the fastest growing mode of the linearization about equilibrium.
- Prove that this growing mode is “nonlinearly stable”.

Difficulty: Precession

Due to precession, the linearization is not self-adjoint. Recall:

\[
J \left( \partial_t \omega + (u \cdot \nabla) \omega \right) + \omega \times J \omega \\
= \tau e_3 + \kappa \nabla \times u - 2\kappa \omega + (\alpha - \gamma) \nabla (\nabla \cdot \omega) + \gamma \Delta \omega.
\]
Precession

Micropolar fluid:

\[ J (\partial_t \omega + (u \cdot \nabla) \omega) + \omega \times J \omega = \tau e_3 + \kappa \nabla \times u - 2\kappa \omega + (\alpha - \gamma) \nabla (\nabla \cdot \omega) + \gamma \Delta \omega. \]

Freely rotating rigid body with inertia \( J \) and angular velocity \( \theta \):

\[ \frac{d}{dt} (J \theta) = J \frac{d}{dt} \theta + \theta \times J \theta = 0. \]
Unstable case - Spectral analysis

The linearization about equilibrium is \( \mathcal{L}X = 0 \), on \( \mathbb{T}^3 \), where the unknown is \( X = (u, \omega, J) \).

We study the spectrum of \( \hat{\mathcal{L}}(k) \) for \( k \in \mathbb{Z}^3 \). For large \( |k| \):

\[
\text{Re} \lambda(k) \to 0 \quad \text{as} \quad |k| \to \infty
\]
The role of anisotropy – Stable case (cartoon)

Consider, where \( a = (J_{13}, J_{23}) \) and \( \chi > 0 \), this cartoon of the linearization

\[
\begin{align*}
\partial_t \omega &= -\omega + \Delta \omega + a \\
\partial_t a &= -\chi \omega
\end{align*}
\]

Solutions satisfy the energy-dissipation relation

\[
\frac{d}{dt} \int_{T^3} \frac{1}{2} |\omega|^2 + \int_{T^3} \frac{1}{2} \chi |a|^2 = -\int_{T^3} |\omega|^2 + |\nabla \omega|^2.
\]

Bootstrapping (formally):

\[
\mathcal{E} = ||\omega||_{L^2}^2 + ||a||_{L^2}^2 + ||\partial_t \omega||_{L^2}^2 + ||\partial_t a||_{L^2}^2
\]

\[
\mathcal{D} = ||\omega||_{H^1}^2 + ||\partial_t \omega||_{H^1}^2 \geq ||a||_{H^{-1}}^2 + ||\partial_t a||_{H^1}^2
\]
Stable case – Hypo-coercivity

Recall:

\[ E = \| \omega \|_{L^2}^2 + \| a \|_{L^2}^2 + \| \partial_t \omega \|_{L^2}^2 + \| \partial_t a \|_{L^2}^2 \]

\[ D = \| \omega \|_{H^1}^2 + \| \partial_t \omega \|_{H^1}^2 \gtrsim \| a \|_{H^{-1}}^2 + \| \partial_t a \|_{H^1}^2 \]

Hypo-coercivity

By interpolation (formally):

\[ \| a \|_{L^2}^2 \lesssim \| a \|_{H^{-1}}^{2\theta} \| a \|_{H^s}^{2(1-\theta)} \]

where \( \theta = \frac{s}{1+s} \uparrow 1 \) as \( s \uparrow \infty \). Therefore

\[ E \lesssim D^\theta E_{\text{high}}^{1-\theta} \Rightarrow E(t) \lesssim \frac{E(0)}{(1+t)^\alpha}, \quad \alpha = \frac{\theta}{1-\theta} \uparrow \infty \text{ as } \theta \uparrow 1. \]
Thank you for your attention!