Decay of surface waves

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Physics and setup

Goal

Energy-dissipation relation

Closing the estimates
**Setup and PDE**

**Unknowns:** velocity $u$, pressure $p$ and surface elevation $\eta$

**Physical constants:** fluid density $\rho$ and gravity $g$.

\[
\begin{align*}
\rho \frac{\partial}{\partial t} u &= \nabla \cdot (\nabla u - pl) & \text{in } \Omega(t) \\
\nabla \cdot u &= 0 & \text{in } \Omega(t) \\
pl - \nabla u \cdot \nu &= \left( \Delta^2 \eta + \rho g \eta \right) \nu & \text{on } \Sigma(t) \\
\frac{\partial}{\partial t} \eta &= (u \cdot \nu) \sqrt{1 + |\nabla \eta|^2} & \text{on } \Sigma(t) \\
u &= 0 & \text{on } \Sigma_b
\end{align*}
\]

where $\nabla \cdot = \text{tr } \nabla$ and $\Delta = \text{tr } \nabla^2$. 
Surface energy: Where does $\Delta^2 \eta$ come from?

Suppose that we have a ‘bending’ energy

$$ \int_{\Sigma(t)} \frac{1}{2} H^2 \sim \int_{\mathbb{T}^2} \frac{1}{2} |\Delta \eta|^2 $$

where $H = 0, H = 0, H = 1$.

Variations of a functional:

$$ \delta \phi \mathcal{F}(\eta) := \frac{d}{dt} \mathcal{F}(\eta + t\phi)|_{t=0} $$

$$ \delta \phi \int_{\mathbb{T}^2} \frac{1}{2} |\Delta \eta|^2 = \int_{\mathbb{T}^2} (\Delta \eta)(\delta \phi \Delta \eta) = \int_{\mathbb{T}^2} \Delta \eta \Delta \phi = \int_{\mathbb{T}^2} (\Delta^2 \eta) \phi $$

(c.f. $\int fg'' = -\int f'g' = \int f''g$).

We call $\Delta^2 \eta$ the first variation of $\int_{\mathbb{T}^2} \frac{1}{2} |\Delta \eta|^2$. 
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Exponential decay to equilibrium, i.e. \((u, \eta) \to 0\) exponentially fast, in some norm, as \(t \to \infty\).
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Physics and setup

Goal

Energy-dissipation relation

Closing the estimates
Energy-dissipation relation

Recall

\[
\left\{ \begin{array}{ll}
\rho \partial_t u = \nabla \cdot (\nabla u - pl) & \text{in } \Omega(t) \\
\nabla \cdot u = 0 & \text{in } \Omega(t) \\
(pl - \nabla u) \cdot \nu = (\Delta^2 \eta + \rho g \eta) \nu & \text{on } \Sigma(t) \\
\partial_t \eta = (u \cdot \nu) \sqrt{1 + |\nabla \eta|^2} & \text{on } \Sigma(t) \\
u = 0 & \text{on } \Sigma_b
\end{array} \right.
\]

Multiply the first PDE by $u$ and integrate by parts to obtain

\[
\frac{d}{dt} \left( \int_{\Omega(t)} \frac{1}{2} \rho |u|^2 + \int_{\Sigma(t)} |\Delta \eta|^2 + \int_{\Sigma(t)} \frac{1}{2} \rho g |\eta|^2 \right) + \int_{\Omega(t)} |\nabla u|^2 = 0
\]
Physics and setup

Goal

Energy-dissipation relation

Closing the estimates
Poincaré inequality

In the absence of a free surface, the energy-dissipation relation becomes (WLOG \( \rho = 1 \))

\[
\frac{d}{dt} \int_{\Omega(t)} |u|^2 + \int_{\Omega(t)} |\nabla u|^2 = 0
\]

In particular, since \( u = 0 \) on \( \Sigma_b \), Poincaré tells us that

\[
\int_{\Omega(t)} |u|^2 \leq C \int_{\Omega(t)} |\nabla u|^2
\]

Therefore we have the differential inequality

\[
\frac{d}{dt} \left( E(t) e^{t/C} \right) = \frac{d}{dt} E(t) + \frac{1}{C} E(t) \leq \frac{d}{dt} E(t) + D(t) = 0
\]

So finally, upon integrating: \( E(t) \leq E(0) e^{-t/C} \)
Recall the dynamic boundary condition

\[(pl - \nabla u) \cdot \nu = (\Delta^2 \eta + \rho g \eta) \nu \quad \text{on } T^2\]

Embedded in there is the PDE

\[\Delta^2 \eta = f \quad \text{on } T^2\]

for \(f = [(pl - \nabla u) \cdot \nu] \cdot \nu - \rho g \eta\).

Taking the Fourier transform (i.e. writing the PDE in an orthonormal basis diagonalizing differential operators)

\[\frac{1}{(2\pi)^4} |k|^4 \hat{\eta}(k) = \hat{f}(k) \quad \text{for all } k \in \mathbb{Z}^2\]
Relating $u$ and $\eta$: an elliptic estimate

Suppose $\eta$ solves $\Delta^2 \eta = f$, which on the Fourier side reads

$$(2\pi)^4 |k|^4 \hat{\eta}(k) = \hat{f}(k) \quad \text{for all } k \in \mathbb{Z}^2$$

Therefore

$$\int_{\mathbb{T}^2} |\nabla^4 \eta|^2 = \sum_{k \in \mathbb{Z}^2} \left( |k|^4 |\hat{\eta}(k)| \right)^2 = \frac{1}{(2\pi)^8} \sum_{k \in \mathbb{Z}^2} |\hat{f}(k)|^2 = \frac{1}{(2\pi)^8} \int_{\mathbb{T}^2} |f|^2$$

In particular, for $f = (\nabla u \cdot \nu) \cdot \nu$,

$$\int_{\mathbb{T}^2} |\nabla^4 \eta|^2 \leq C \int_{\mathbb{T}^2} |\nabla u|^2 \sim \int_{\Sigma(t)} |\nabla u|^2 = \int_{\partial \Omega(t)} |\nabla u|^2$$
Trace inequality

The energy-dissipation relation gives us control of $\int_{\Omega(t)} |\nabla u|^2$
and to close the estimates we need control of $\int_{\partial \Omega(t)} |\nabla u|^2$.

However

$$\int_{\partial \Omega(t)} f^2 \leq C \int_{\Omega(t)} f^2$$

Trace inequality:

$$\int_{\partial \Omega(t)} f^2 \leq C \left( \int_{\Omega(t)} f^2 + \int_{\Omega(t)} |\nabla f|^2 \right)$$
Trace inequality (1D)

Consider $f : (0, 1) \to \mathbb{R}$. Observe that $f(1) = \int_0^1 (xf(x))'$. 

$$|f(1)| = \left| \int_0^1 (xf(x))' \right|$$

$$\leq \int_0^1 |f(x)| + \int_0^1 |xf'(x)|$$

$$\leq \left( \int_0^1 1 \right)^{1/2} \left( \int_0^1 f(x)^2 \right)^{1/2} + \left( \int_0^1 x^2 \right)^{1/2} \left( \int_0^1 f'(x)^2 \right)^{1/2}$$

$$= \left( \int_0^1 f(x)^2 \right)^{1/2} + \frac{1}{\sqrt{3}} \left( \int_0^1 f'(x)^2 \right)^{1/2}$$
Closing the estimates

Differentiate the PDE, multiply by $u$, and integrate by parts

\[
\frac{d}{dt} \left( \int_{\Omega(t)} \frac{1}{2} \rho |u|^2 + \int_{\Sigma(t)} |\Delta \eta|^2 + \int_{\Sigma(t)} \frac{1}{2} \rho g |\eta|^2 \right) + \int_{\Omega(t)} |\nabla u|^2 \\
+ \frac{d}{dt} \left( \int_{\Omega(t)} \frac{1}{2} \rho |\nabla u|^2 + \int_{\Sigma(t)} |\Delta \nabla \eta|^2 + \int_{\Sigma(t)} \frac{1}{2} \rho g |\nabla \eta|^2 \right) + \int_{\Omega(t)} |\nabla^2 u|^2 = 0
\]

Dynamic boundary condition: $\Delta^2 \eta \sim \nabla u$ on $\Sigma(t)$

\[
\int_{\Sigma(t)} |\nabla^4 \eta|^2 \leq C \int_{\Sigma(t)} |\nabla u|^2 \leq C \left( \int_{\Omega(t)} |\nabla u|^2 + \int_{\Omega(t)} |\nabla^2 u|^2 \right)
\]

i.e. indeed, using Poincaré's inequality, \( \frac{d}{dt} E + CE \leq 0 \) and hence

\[
E(t) \leq E(0) e^{-Ct}
\]
Bonus round!
The fully nonlinear problem

\[
\begin{cases}
\rho \left( \partial_t u + u \cdot \nabla u \right) = \nabla \cdot ( \mathbb{D} u - p I ) & \text{in } \Omega(t) \\
\nabla \cdot u = 0 & \text{in } \Omega(t) \\
(pl - \mathbb{D} u) \cdot \nu = (\delta \mathcal{W}(\eta) + \rho g \eta) \nu & \text{on } \Sigma(t) \\
\partial_t \eta = (u \cdot \nu) \sqrt{1 + |\nabla \eta|^2} & \text{on } \Sigma(t) \\
u = 0 & \text{on } \Sigma_b
\end{cases}
\]

e.g. \( \delta \mathcal{W}(\eta) = \Delta \Sigma(t) H - (H^2 - 4K) \) if \( \mathcal{W}(\eta) = \int_{\Sigma(t)} H^2. \)

Commutators are evil.
Thank you for your attention!