The Viscous Surface Wave Problem with Generalized Surface Energies

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The fluid is three-dimensional, incompressible, and viscous.
We work in a horizontally periodic domain with finite depth.
The upper boundary is free and given as a graph.
Gravity and surface forces act on the fluid.

Unkowns: the velocity $u$, the pressure $p$, and the surface profile $\eta$. 
Surface forces and surface energies

Typical example: surface tension.

\[ \mathcal{E} = \int_{\Sigma(t)} dA = \int_{\mathbb{T}^2} \sqrt{1 + |\nabla \eta|^2} \rightarrow \mathcal{F} = -H \nu \]

Elasticity: Helfrich introduced the Willmore energy

\[ \int_{\Sigma(t)} \frac{1}{2} H^2 dA. \]
Surface forces and surface energies - illustrative example

Throughout this talk we use the Willmore energy as an illustrative example:

$$\mathcal{W}(\eta) = \int_{\Sigma(t)} \frac{1}{2}H^2 dA.$$ 

It gives rise to the surface force

$$\mathcal{F}(\eta) = \delta \mathcal{W}(\eta) = \Delta_{\Sigma(t)} H + \frac{1}{2}H(H^2 - 4K),$$

where, for any test function $\phi$,

$$\int_{\mathbb{T}^2} \delta \mathcal{W}(\eta) \phi := \frac{d}{dt} \mathcal{W}(\eta + t \phi) \bigg|_{t=0}.$$ 

Key feature: $\delta \mathcal{W}$ linearizes (around zero) to the bi-Laplacian $\Delta^2$. 
Exponential asymptotic stability:
Solutions \textit{initially close to equilibrium} (i.e. a quiescent slab) exist globally in time and \textit{approach that equilibrium exponentially fast}. 
The PDE

Why small data?

Our only hope

Strategy

Zoo of energies

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The PDE

\begin{align*}
\partial_t u + (u \cdot \nabla) u &= -\nabla p + \Delta u \quad \text{in } \Omega (t) \\
\nabla \cdot u &= 0 \quad \text{in } \Omega (t) \\
(\rho I - D u) \nu &= (\delta \mathcal{W} (\eta) + g \eta) \nu \quad \text{on } \Sigma (t) \\
\partial_t \eta &= (u \cdot \nu) \sqrt{1 + |\nabla \eta|^2} \quad \text{on } \Sigma (t) \\
u &= 0 \quad \text{on } \Sigma_b
\end{align*}
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Why small data?

Technical considerations:

• Guarantees that graphs remain graphs.
• Avoids splash singularities.

Physical consideration:

• Geometric evolution equations typically lead to singularities in finite-time.
The PDE

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Why is there any hope?

Because we can integrate by parts!

\[
\begin{aligned}
\partial_t u + (u \cdot \nabla) u &= -\nabla p + \Delta u \quad \text{in } \Omega (t) \\
\nabla \cdot u &= 0 \quad \text{in } \Omega (t) \\
(p l - D u) \nu &= (\delta W (\eta) + g \eta) \nu \quad \text{on } \Sigma (t) \\
\partial_t \eta &= (u \cdot \nu) \sqrt{1 + |\nabla \eta|^2} \quad \text{on } \Sigma (t) \\
u &= 0 \quad \text{on } \Sigma_b
\end{aligned}
\]

Multiply the first equation by \( u \) and integrate by parts to obtain

\[
\frac{d}{dt} \left( \int_{\Omega(t)} \frac{1}{2}|u|^2 + W (\eta) + \int_{T^2} \frac{g}{2}|\eta|^2 \right) = - \int_{\Omega(t)} \frac{1}{2}|D u|^2
\]
Leveraging the energy-dissipation relation

Recall

\[
\begin{cases}
(p I - D u) \nu = (\delta \mathcal{W} + g) (\eta) \nu & \text{on } \Sigma (t) \\
u = 0 & \text{on } \Sigma_b
\end{cases}
\]

and

\[
\frac{d}{dt} \left( \int_{\Omega(t)} \frac{1}{2} |u|^2 + \mathcal{W} (\eta) + \int_{\mathbb{T}^2} \frac{g}{2} |\eta|^2 \right) = - \int_{\Omega(t)} \frac{1}{2} |D u|^2.
\]

Bootstrap: (formally)

\[
||D u||_{L^2(\Omega)} \gtrsim ||D u||_{H^{-1/2}(\Sigma)} \gtrsim \left\| (\delta \mathcal{W} + g) (\eta) \right\|_{H^{-1/2}(\Sigma)} \\
\gtrsim ||\eta||_{H^{7/2}(\Sigma)} \gtrsim \mathcal{W} (\eta) + \int \frac{g}{2} |\eta|^2
\]
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Strategy

To obtain exponential decay of the energy we

- Flatten and differentiate

\[ \nabla \text{ in } \Omega(t) \quad \rightarrow \quad \Phi^{-1} \]

\[ \nabla^{G(\eta)} \text{ in } \Omega \]

where \( G(\eta) = (\nabla \Phi)^{-1} \sim I - \nabla \eta \otimes e_3 \) since \( \Phi \sim \text{id} + \eta \ e_3 \).

- Find two perturbative forms which allows us to
  1. Make rigorous the formal bootstrap argument: \( E \lesssim D \).
  2. Control the commutators and close the estimates:
     \[
     \frac{d}{dt} E + D = C \quad \text{with} \quad |C| \lesssim \sqrt{ED}.
     \]
Finding a good perturbative form - cartoon

We seek to write $\partial^\alpha (\delta \mathcal{W} (\eta) = \ldots )$ as $\mathcal{L}_\eta (\partial^\alpha \eta) = C^\alpha$ such that

- the commutators $C^\alpha$ can be controlled and
- the operator $\mathcal{L}_\eta$ has good integration by parts.

A good choice is the second variation $\mathcal{L}_\eta = \delta^2_\eta \mathcal{W}$. We then have

- good commutators since $\partial^\alpha (\delta \mathcal{W} (\eta)) = (\delta^2_\eta \mathcal{W}) \partial^\alpha \eta$ and
- good integration by parts since $\int_{\mathbb{T}^2} ((\delta^2_\eta \mathcal{W}) \phi) \phi = Q_\eta (\phi)$

where $Q$ is the quadratic approximation of $\mathcal{W}$ about $\eta$.

\[ \mathcal{W} \quad \xrightarrow{\text{approx.}} \quad Q_\eta \]
\[ \delta \mathcal{W} \quad \xrightarrow{\text{approx.}} \quad \delta^2_\eta \mathcal{W} \]

‘It’s the energy, stupid!’
Theorem (exponential asymptotic stability). If $\mathcal{E}_0$ is sufficiently small then $\mathcal{E}(t) \leq \mathcal{E}_0 e^{-\lambda t}$, for some $\lambda > 0$, where

$$\mathcal{E} := \|u\|_{H^2} + \|\partial_t u\|_{L^2} + \|p\|_{H^1} + \|\eta\|_{H^{9/2}} + \|\partial_t \eta\|_{H^2}.$$
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Zoo of energies
A zoo of surface energies

In this talk: \( \mathcal{W}_{\text{Will.}}(\eta) = \int_{\mathbb{T}^2} H^2 dA, \ \delta_0^2 \mathcal{W}_{\text{Will.}} = \Delta^2. \)

We may also consider

- **Lower order terms:**
  \[ \mathcal{W}_{\text{Helfrich}} = \int_{\mathbb{T}^2} (C_1 + C_2 H^2) dA, \]
  \[ \delta_0^2 \mathcal{W}_{\text{Helfrich}} = -C_1 \Delta + C_2 \Delta^2. \]

- **Anisotropies:**
  \[ \mathcal{W}_{\text{anis.}} = \int_{\mathbb{T}^2} \frac{1}{2} \left| C (\nabla \eta) : \nabla^2 \eta \right|^2 \text{ with } C(0) > 0, \]
  \[ \delta_0^2 \mathcal{W}_{\text{anis.}} = (C(0) : \nabla^2)^2. \]

Key assumption of our theorem: \( \delta_0^2 \mathcal{W} + g \) is strictly elliptic (over functions of average zero).

Also allows:

\[ \downarrow g \]
Thank you for your attention!