Viscous Surface Wave Problem with Generalized Surface Energies

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Goal

Setup: first there was physics

Difficulties: nonlinearities and competition

Resolution: the energy-dissipation relation
The goal is to prove

- well-posedness and
- exponential decay

of the viscous wave problem with generalized surface energies *for small initial data*. 
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What is the viscous surface wave problem?

Unkowns: velocity $u$, pressure $p$ and surface elevation $\eta$.
Physical constants: fluid density $\rho$ and gravity $g$.

\[
\begin{aligned}
\rho \left( \partial_t u + (u \cdot \nabla) u \right) &= \nabla \cdot (\nabla u - pl) & \text{in } \Omega(t) \\
\nabla \cdot u &= 0 & \text{in } \Omega(t) \\
(pI - \nabla u) \cdot \nu &= (\Delta^2 \eta + \rho g \eta ) \nu & \text{on } \Sigma(t) \\
\partial_t \eta &= (u \cdot \nu) \sqrt{1 + |\nabla \eta|^2} & \text{on } \Sigma(t) \\
u &= 0 & \text{on } \Sigma_b
\end{aligned}
\]
Surface energy: Where does $\Delta^2 \eta$ come from?

Suppose that we have a ‘bending’ energy $\int_{\Sigma(t)} \frac{1}{2} H^2 \sim \int_{\mathbb{T}^2} \frac{1}{2} |\Delta \eta|^2$

Then

$$\frac{d}{dt} \int_{\mathbb{T}^2} \frac{1}{2} |\Delta (\eta + t\phi)|^2 \bigg|_{t=0} = \int_{\mathbb{T}^2} \Delta \eta \Delta \phi = \int_{\mathbb{T}^2} (\Delta^2 \eta) \phi.$$
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Why is this a hard problem?

- Technical trade-off: fix the domain at the expense of worse nonlinearities.

- Competing physical and mathematical effects:

  \[ \rho \left( \partial_t u + (u \cdot \nabla) u \right) = \nabla \cdot (\nabla u - pl) \quad \text{(parabolic)} \]

  \[ (pl - \nabla u) \cdot \nu = (\Delta^2 \eta + \rho g \eta) \nu \quad \text{(elliptic)} \]

  \[ \partial_t \eta = (u \cdot \nu) \sqrt{1 + |\nabla \eta|^2} \quad \text{(hyperbolic)} \]
### Goal

**Setup:** first there was physics

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**Resolution:** the energy-dissipation relation
How do we solve this?

‘It’s the energy-dissipation relation, stupid!”

Recall

\[
\begin{cases}
\rho (\partial_t u + (u \cdot \nabla) u) = \nabla \cdot (\nabla u - pl) & \text{in } \Omega(t) \\
\nabla \cdot u = 0 & \text{in } \Omega(t) \\
(pl - \nabla u) \cdot \nu = (\Delta^2 \eta + \rho g \eta) \nu & \text{on } \Sigma(t) \\
\partial_t \eta = (u \cdot \nu) \sqrt{1 + |\nabla \eta|^2} & \text{on } \Sigma(t) \\
u = 0 & \text{on } \Sigma_b
\end{cases}
\]

We multiply the first PDE by \( u \) and integrate by parts to obtain

\[
\frac{d}{dt} \left( \int_{\Omega(t)} \frac{1}{2} \rho |u|^2 + \int_{\Sigma(t)} |\Delta \eta|^2 + \int_{\Sigma(t)} \frac{1}{2} \rho g |\eta|^2 \right) + \int_{\Omega(t)} |\nabla u|^2 = 0
\]
The energy-dissipation relation saves the day

Schematically

\[ \frac{d}{dt} \left( \int_{\Omega(t)} |u|^2 + \int_{\Sigma(t)} |\eta|^2 + |\Delta \eta|^2 \right) + \int_{\Omega(t)} |\nabla u|^2 = 0 \]

The dynamic boundary condition tells us that \( E(\eta) \leq C_1 E(u) \).

The no-slip boundary conditions tells us that \( E(u) \leq C_2 D(u) \).

Putting it together: \( \mathcal{E} \leq CD(u) \), i.e.

\[ \mathcal{E} := E(u) + E(\eta) \leq (1 + C_1)E(u) \leq (1 + C_1)C_2 D(u) =: CD(u). \]

So finally

\[ \frac{d}{dt} \mathcal{E} + \frac{1}{C} \mathcal{E} \leq \frac{d}{dt} \mathcal{E} + D = 0 \Rightarrow \mathcal{E}(t) \leq e^{-\frac{t}{C}} \mathcal{E}(0). \]
Thank you for your attention!