Recitation Handout 14: Review of Geometric Series

There is a particular kind of series called a **geometric series** that has a lot of very useful applications – mortgage calculations, for example, which we will be doing today. This kind of series can be recognized by the pattern formed by the terms:

$$a + a{\cdot}r + a{\cdot}r^2 + ... + a{\cdot}r^n.$$

- The number a is referred to as the **initial value**.

- The number r is referred to as the **multiplicative factor**.

- One of the distinguishing features of a geometric series is that the ratio of successive terms is constant (it is equal to r).

The sum of a geometric series can be calculated with a relatively straightforward summation formula.

$$a + a \cdot r + a \cdot r^2 + ... + a \cdot r^n = \frac{a \cdot (1 - r^{n+1})}{1 - r}.$$

The exponent that appears in the summation formula is **always** the number of terms that are being added together in the geometric series.

Infinite Geometric Series

An infinite geometric series is a geometric series:

$$a + a{\cdot}r + a{\cdot}r^2 + a{\cdot}r^3 + a{\cdot}r^4 + ... + a{\cdot}r^k + ...$$

that never stops. The fact that the series never stops is indicated by avoiding a "final term" when writing down the series, and ending the series with an ellipsis (…).

An infinite geometric series consists of an infinitely long string of terms that are added together.

The Sum of an Infinite Geometric Series

What is the sum of an infinite geometric series? You might intuitively think that because infinitely many numbers are added together in an infinite series that the total would *have* to be infinite. Often that is the case, but amazingly sometimes it is possible to add up an infinite list of numbers and get a *finite* total.

An geometric series is a series with the following format:

$$a + a\cdot r + a\cdot r^2 + a\cdot r^3 + a\cdot r^4 + \ldots + a\cdot r^n + \ldots$$

What the sum of the first n terms? The first n terms of an infinite geometric series looks just like the finite geometric series that we are accustomed to.

$$\textit{First "n" terms} = a + a\cdot r + a\cdot r^2 + a\cdot r^3 + a\cdot r^4 + \ldots + a\cdot r^{n-1}$$

You can add this up with the summation formula for a geometric series.

$$\textit{Sum of the first "n" terms} = \frac{a\cdot(1-r^n)}{1-r}.$$

What would you do to "n" so that the sum would become the sum of the whole infinite series? Answer: Let $n \to \infty$ and figure out the limit of the geometric summation formula.

$$\textit{Sum of the first "n" terms} = \frac{a\cdot(1-r^n)}{1-r}.$$

Computing the Sum of an Infinite Geometric Series

The only part of the geometric summation formula that is directly affected by n is the power r^n that appears in the numerator. The limit as $n \to \infty$ will depend on the value of the multiplicative constant r, and there are two main situations to consider.

Case 1: $-1 < r < 1$ In this case, $r^n \to 0$ as $n \to \infty$.
                                         The sum of the infinite geometric
                                         series is the limit:

                                         $\boxed{}$.

                                         In the next few classes we will begin
                                         to describe an infinite series that has
                                         a finite sum as a **convergent** series.

Examples:

(a)   The sum of the infinite geometric series ($a = 1$ and $r = \frac{1}{2}$):

$$1 + \tfrac{1}{2} + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^3 + \ldots + \left(\tfrac{1}{2}\right)^n + \ldots$$

is equal to $\dfrac{1}{1-\frac{1}{2}} = 2.$

(b)   The sum of the infinite geometric series ($a = 1$ and $r = \frac{-1}{2}$):

$$1 + \tfrac{-1}{2} + \left(\tfrac{-1}{2}\right)^2 + \left(\tfrac{-1}{2}\right)^3 + \ldots + \left(\tfrac{-1}{2}\right)^n + \ldots$$

is equal to $\dfrac{1}{1 - \frac{-1}{2}} = \dfrac{2}{3}$.

Case 2: $r \le -1$ or $r \ge 1$ In this case the summation formula does not have a limit and we say that the infinite geometric series does not have a sum.

In the next few classes we will begin to describe an infinite series that does not have a finite sum as a **divergent** series.

Examples:

(a)   The sum of the infinite geometric series ($a = 1$ and $r = 2$):

$$1 + 2 + 2^2 + 2^3 + \ldots + 2^n + \ldots$$

is not a finite number. This isn't a big surprise as the series adds together an infinite list of numbers that just get bigger and bigger.

(b)   It is not even really possible to say that the sum of the infinite geometric series ($a = 1$ and $r = -1$):

$$1 + (-1) + (-1)^2 + (-1)^3 + \ldots + (-1)^n + \ldots$$

is a number at all. If you work out what all of the powers of -1 are, the infinite series looks like this:

$$1 - 1 + 1 - 1 + \ldots + (-1)^n + \ldots$$

Each "pair" of a "1" and a "−1" will cancel out. (Remember the series at the end of the calculator handout.) If this series was a finite series, then the sum of the series would either be "1" (if the last term was an unmatched "1") or zero (if the last term was a "−1"). However, as this is an infinite series, it never ends and so it has no last term. Therefore, it is impossible to say what the sum of the series is, and instead we simply say that there is no sum for this series.