Asymptotic behavior of Laplacian eigenvalues of subspace inclusion graphs

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Introduction

- $\mathbb{F}_q^n$ = n-dimensional vector space over the field of q elements.
- $\binom{n}{k}_q$ = number of k-dimensional subspaces of $\mathbb{F}_q^n$
  $= \frac{\prod_{i=n-k+1}^{n}(q^i-1)}{\prod_{i=1}^{k}(q^i-1)}$.
- $\mathcal{S}_{n,q}$ = set of all non-trivial subspaces of $\mathbb{F}_q^n$.
- $\Delta_{n,q}$ = a $\mathcal{S}_{n,q} \times \mathcal{S}_{n,q}$ matrix defined by:

$$(\Delta_{n,q})_{U,V} = \begin{cases} n-2 & \text{if } U = V, \\ -\binom{n-\dim(U)}{\dim(V)-\dim(U)}_q^{-1} & \text{if } U \subsetneq V, \\ -\binom{\dim(U)}{\dim(V)}_q^{-1} & \text{if } V \subsetneq U, \\ 0 & \text{otherwise.} \end{cases}$$

for all $U, V \in \mathcal{S}_{n,q}$.

Introduction

- We can think of $\Delta_{n,q}$ as a weighted Laplacian matrix associated to the graph

$$G_{n,q} = \left(\mathcal{S}_{n,q}, \{\{U, V\} : U \subsetneq V \text{ or } V \subsetneq U\}\right).$$

- Our goal: estimate the eigenvalues of $\Delta_{n,q}$ (for fixed $n \geq 3$ and large q).

Motivation

- High dimensional Laplacians on simplicial complexes
- Complexes of flags (i.e. spherical buildings)
- A conjecture of Papikian.

Simplicial complexes

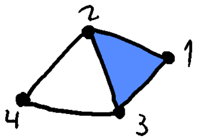
Let V be a finite set.
A family of subsets $X \subset 2^V$ is called a simplicial complex if it satisfies:

$$A \in X \text{ and } B \subset A \implies B \in X.$$

- An element $A \in X$ is called a simplex (or face) of X.
- The dimension of a simplex A is $|A| - 1$.
- The dimension of $X = \max_{A \in X} \dim(A)$.

We may think of a simplicial complex as a geometric object:

$$X = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\},\\ \{1,2\}, \{1,3\}, \{2,3\},\\ \{2,4\}, \{3,4\}, \{1,2,3\}\}$$



High dimensional Laplacians

- Let $k \geq -1$. Define

$$X(k) = \{\sigma \in X : \dim(\sigma) = k\}.$$

Space of *k-cochains*:

$$C^k(X) = \{\phi : X(k) \to \mathbb{R}\} = \mathbb{R}^{X(k)}.$$

- Inner product on $C^k(X)$. For $w : X \to \mathbb{R}_{>0}$,

$$\langle \phi, \psi \rangle = \sum_{\sigma \in X(k)} w(\sigma)\phi(\sigma)\psi(\sigma).$$

High dimensional Laplacians

- Coboundary operator $d_k : C^k(X) \to C^{k+1}(X)$:
  For $\phi \in C^k(X)$,

$$d_k\phi([v_0, \dots, v_{k+1}]) = \sum_{i=0}^{k+1} (-1)^i \phi([v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_{k+1}]).$$

- $d_k^* : C^{k+1}(X) \to C^k(X)$:

$$\langle d_k\phi, \psi\rangle = \langle \phi, d_k^*\psi\rangle$$

- $\Delta_k^+(X) : C^k(X) \to C^k(X) =$ k-dimensional Laplacian.

$$\Delta_k^+(X) = d_k^* d_k.$$

High dimensional Laplacians – Matrix form

$\Delta_k^+(X)$ is an $X(k) \times X(k)$ matrix, with elements

$$\Delta_k^+(X)_{\sigma,\tau} = \begin{cases} \displaystyle\sum_{v \in N_X(\sigma)} \frac{w(\sigma \cup \{v\})}{w(\sigma)} & \text{if } \sigma = \tau, \\ \pm \dfrac{w(\sigma \cup \tau)}{w(\sigma)} & \text{if } \sigma \sim \tau, \\ 0 & \text{otherwise.} \end{cases}$$

In the special case $k = 0$:

$$\Delta_0^+(X)_{u,v} = \begin{cases} \displaystyle\sum_{u' \in N_X(u)} \frac{w(\{u, u'\})}{w(\{u\})} & \text{if } u = v, \\ -\dfrac{w(\{u, v\})}{w(\{u\})} & \text{if } \{u, v\} \in X, \\ 0 & \text{otherwise.} \end{cases}$$

Complexes of flags

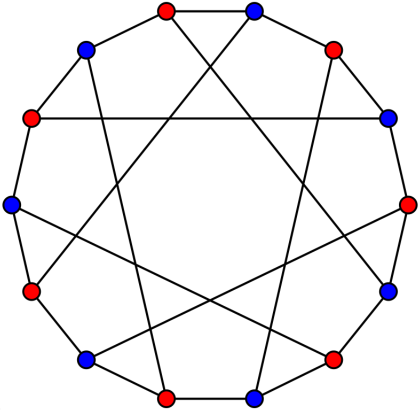
- Recall: $\mathcal{S}_{n,q}$ = set of all non-trivial subspaces of $\mathbb{F}_q^n$.
- A flag is a family of subspaces $\{V_1, \ldots, V_k\}$ such that

$$V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_k.$$

- $\mathrm{Fl}_{n,q}$ = simplicial complex on vertex set $\mathcal{S}_{n,q}$ whose simplices are the flags in $\mathbb{F}_q^n$.

Complexes of flags- Example

The complex $\mathrm{Fl}_{3,2}$:



(Source: Wikipedia)

A weight function on the complex of flags

- A complete flag is a flag of the form $V_1 \subset V_2 \subset \cdots \subset V_{n-1}$, where $\dim(V_i) = i$ for all i.
- $\mathrm{Fl}_{n,q}$ is a pure $(n-2)$-dimensional complex. (all maximal faces are of dimension $n-2$: the complete flags).
- For $\sigma = \{V_1, \ldots, V_k\} \in \mathrm{Fl}_{n,q}$, we define

$$\begin{aligned} w(\sigma) &= \text{ number of maximal faces of } \mathrm{Fl}_{n,q} \text{ containing } \sigma \\ &= \text{ number of complete flags extending } \sigma. \end{aligned}$$

Eigenvalues of $\Delta_k^+(\mathrm{FI}_{n,q})$

- The multiplicity of 0 as an eigenvalue of $\Delta_k^+(\mathrm{FI}_{n,q})$ is determined by the homology groups of $\mathrm{FI}_{n,q}$, and is well understood.

Theorem (Garland '73)

Let $n \geq 3$, $0 \leq k \leq n-3$. Let $\epsilon > 0$.
Then, there exists $q_0 = q_0(n, \epsilon)$ such that for $q \geq q_0$, every non-zero eigenvalue λ of $\Delta_k^+(FI_{n,q})$ satisfies

$$\lambda \geq n - k - 2 - \epsilon.$$

Papikian's conjecture

Conjecture (Papikian '16)

- The number of distinct eigenvalues of $\Delta_k^+(\mathrm{Fl}_{n,q})$ does not depend on q.
- Let $\epsilon > 0$. Then, there exists $q_0 = q_0(n, \epsilon)$ such that for $q \geq q_0$, every non-zero eigenvalue of $\Delta_k^+(\mathrm{Fl}_{n,q})$ is ϵ-close to one of the integers $n-k-2, n-k-1, \ldots, n-1$.
  For $k = 0$: All non-zero eigenvalues of $\Delta_0^+(\mathrm{Fl}_{n,q})$ tend to $n-2$ or $n-1$.

Main result

Recall:

- $\mathcal{S}_{n,q}$ = set of all non-trivial subspaces of $\mathbb{F}_q^n$.
- $\Delta_{n,q}$ = a $\mathcal{S}_{n,q} \times \mathcal{S}_{n,q}$ matrix defined by:

$$(\Delta_{n,q})_{U,V} = \begin{cases} n-2 & \text{if } U = V, \\ -\binom{n-\dim(U)}{\dim(V)-\dim(U)}_q^{-1} & \text{if } U \subsetneq V, \\ -\binom{\dim(U)}{\dim(V)}_q^{-1} & \text{if } V \subsetneq U, \\ 0 & \text{otherwise.} \end{cases}$$

  for all $U, V \in \mathcal{S}_{n,q}$.

- $\Delta_{n,q}$ is exactly the 0-dimensional Laplacian $\Delta_0^+(\mathrm{Fl}_{n,q})$.

Main result

Theorem (L' 23+)

Let $n \geq 3$, $q \geq q_0(n)$ a prime power. Then, the eigenvalues of $\Delta_{n,q} = \Delta_0^+(Fl_{n,q})$ are:

- 0 *with multiplicity* 1,
- $n-1$ *with multiplicity* $n-2$,
- *For* $1 \leq k \leq \lfloor (n-1)/2 \rfloor$ *and every* ζ *in*

$$\mathcal{J}_k = \left\{ \pm 2\cos\left(\frac{j\pi}{n-2k+2}\right) : 1 \leq j \leq \left\lfloor \frac{n-2k+1}{2} \right\rfloor \right\},$$

$$\lambda \approx n - 2 + \zeta \cdot q^{-k/2}$$

is an eigenvalue with multiplicity $\binom{n}{k}_q - \binom{n}{k-1}_q$.

Main result- continued

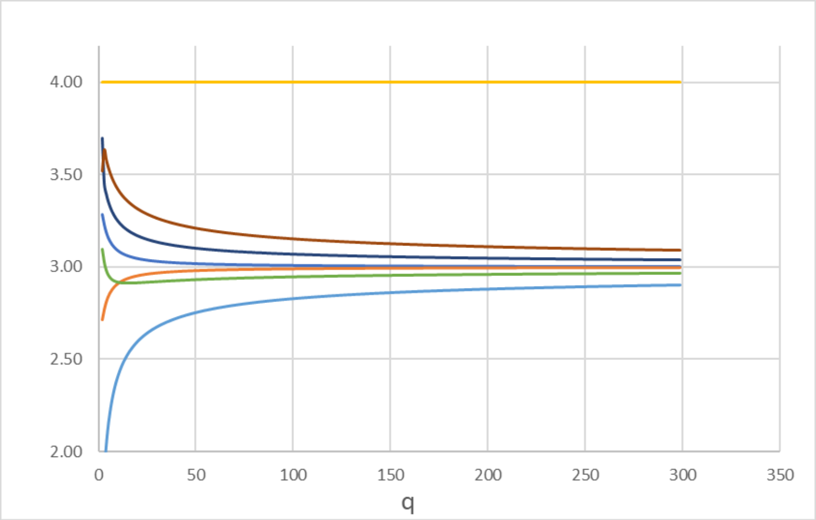
If n is even, we have the additional eigenvalues:

- $n-2$ with multiplicity $\binom{n}{n/2}_q - \binom{n}{n/2-1}_q$,
- For $1 \leq k \leq n/2-1$,

$$\lambda \approx n - 2 + \frac{2(n-2k)}{n-2k+2} \cdot q^{-k}$$

with multiplicity $\binom{n}{k}_q - \binom{n}{k-1}_q$.

Example: $n = 5$



Main result - consequences

As an consequence, we obtain:

Corollary (L' 23+)

- There exists $q_0 = q_0(n)$ such that for $q \geq q_0$, the number of distinct eigenvalues of $\Delta_{n,q}$ is exactly $\left\lfloor \frac{n^2}{4} \right\rfloor + 2$.
- Let $\epsilon > 0$. Then, there exists $q_0 = q_0(n, \epsilon)$ such that for $q \geq q_0$, every eigenvalue $\lambda \neq 0, n-1$ of $\Delta_{n,q}$ satisfies $|\lambda - (n-2)| < \epsilon$.

This solves the 0-dimensional case of Papikian's conjecture (for large q).

Proof sketch

Two main ingredients for proof:

- Find a basis of $C^0(\mathrm{FI}_{n,q})$ in which $\Delta_{n,q}$ has a "nicer" matrix representation (block diagonal with "small" blocks).
- Estimate the eigenvalues by approximating the characteristic polynomial of each block.

Subspace inclusion matrices

- $S(i) = i$-dimensional subspaces of $\mathbb{F}_q^n$.
- For $0 \leq i, j \leq n$, $A_{ij} = S(i) \times S(j)$ matrix

$$(A_{ij})_{U,V} = \begin{cases} 1 & \text{if } U \subset V \text{ or } V \subset U, \\ 0 & \text{otherwise.} \end{cases}$$

- We can write $\Delta_{n,q}$ as an $(n-1) \times (n-1)$ block matrix:

$$\Delta_{n,q} = \begin{pmatrix} L_{1,1} & \cdots & L_{1,n-1} \\ \vdots & & \vdots \\ L_{n-1,1} & \cdots & L_{n-1,n-1} \end{pmatrix},$$

where L_{ij} is an $S(i) \times S(j)$ matrix:

$$L_{ij} = \begin{cases} (n-2)I & \text{if } i = j, \\ -\binom{n-i}{j-i}_q^{-1} A_{ij} & \text{if } i < j, \\ -\binom{i}{j}_q^{-1} A_{ij} & \text{if } i > j. \end{cases}$$

Properties of Subspace inclusion matrices

Theorem (Kantor '72)

- A_{ij} is of full rank.
- Let $k \leq j \leq i$. Then $A_{ij}A_{jk} = \binom{i-k}{j-k}_q A_{ik}$.

Lemma (L' 23+)

Let $k \leq i \leq j$. Then,

$$A_{ij}A_{jk} = \sum_{m=0}^{k} c_{ijkm} A_{im} A_{mk},$$

where $c_{ijkm} = \sum_{r=0}^{m} (-1)^{m-r} q^{\binom{r+1}{2}+\binom{m}{2}-rm} \binom{m}{r}_q \binom{n-i-k+r}{j-i-k+r}_q$.

A change of basis

Idea:

We will choose a basis B of $C^0(\mathrm{Fl}_{n,q})$ consisting of vectors of the form

$$A_{jk} v$$

for $k \leq j \leq n-k$, where v satisfies $A_{ik} v = 0$ for all $0 \leq i < k$.

A change of basis

Theorem (L' 23+)

There is a basis B of $C^0(Fl_{n,q})$ such that the matrix representation of $\Delta_{n,q}$ with respect to the basis B is a block diagonal matrix

$$\begin{pmatrix} L_0 & & \\ & \ddots & \\ & & L_{\lfloor \frac{n}{2} \rfloor} \end{pmatrix}$$

with blocks $L_k = I_{\binom{n}{k}_q - \binom{n}{k-1}_q} \otimes \tilde{L}_k$.

A change of basis

Theorem (L' 23+, continued)

Where $\tilde{L}_0$ is the $(n-1)\times(n-1)$ matrix:

$$(\tilde{L}_0)_{ij} = \begin{cases} n-2 & \text{if } i = j, \\ -1 & \text{if } i \neq j \end{cases}$$

for $1 \leq i, j \leq n-1$,
and for $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$, $\tilde{L}_k$ is the $(n-2k+1)\times(n-2k+1)$ matrix

$$(\tilde{L}_k)_{ij} = \begin{cases} n-2 & \text{if } i = j, \\ -c_{ijkk}\binom{n-i}{j-i}_q^{-1} & \text{if } i < j, \\ -\binom{i-k}{j-k}_q \binom{i}{j}_q^{-1} & \text{if } i > j \end{cases}$$

for $k \leq i, j \leq n-k$.

Estimating the characteristic polynomials

We can write

$$\tilde{L}_k = (n-2)I + M,$$

where

$$M_{ij} = \begin{cases} 0 & \text{if } i = j, \\ -1 + O\left(q^{-1}\right) & \text{if } i < j, \\ -q^{-k(i-j)}\left(1 + O\left(q^{-1}\right)\right) & \text{if } i > j. \end{cases}$$

Lemma (L' 23+)

Let $m = n - 2k + 1$. Let $p(t)$ be the characteristic polynomial of M. Then,

$$p(s \cdot q^{-\frac{k}{2}}) = q^{-\frac{km}{2}}\left(F_m(s) + \text{error term}\right),$$

where $F_m(s) = \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^j \binom{m-j}{j} s^{m-2j}$. (The roots of $F_m(s)$ were computed by Donnelly, Dunkum, Huber and Knupp '21.)


THANKS FOR LISTENING!