21-122 - Week 7, Recitation 2

Section 9.3

31. Find the orthogonal trajectories of the family of curves $y = \frac{k}{x}$. Use a graphing device to draw several members of each family on a common screen.

Solution - The curve $y = \frac{k}{x}$ satisfies the differential equation $\frac{dy}{dx} = -\frac{k}{x^2} = -\frac{1}{x}\frac{k}{x} = -\frac{y}{x}$. The orthogonal trajectories must satisfy the differential equation $\frac{dy}{dx} = \frac{x}{y}$. To solve this differential equation, write

$$\int y\, dy = \int x\, dx \implies \tfrac{1}{2}y^2 = \tfrac{1}{2}x^2 + C \implies y^2 - x^2 = C$$

(in the last step, we replaced $2C$ by C). This is a family of hyperbolas with asymptotes $y = \pm x$. □

39. The differential equation $\frac{dP}{dt} = k(M - P)$ is a model for learning. Here, $P(t)$ measures the performance of someone learning a skill after a training time t, M is the maximum level of performance, and k is a positive constant. Solve this differential equation to find an expression for $P(t)$. What is the limit of this expression as $t \to \infty$?

Solution - Write

$$\begin{aligned}
& \tfrac{dP}{dt} = k(M - P) \\
\implies \quad & \int \tfrac{1}{M-P}\, dP = \int k\, dt \\
\implies \quad & -\ln|M - P| = kt + C \\
\implies \quad & \ln|M - P| = -kt + C \\
\implies \quad & |M - P| = e^C e^{-kt} \\
\implies \quad & M - P = \pm e^C e^{-kt} \\
\implies \quad & P = M \pm e^C e^{-kt}
\end{aligned}$$

As $t \to \infty$, we have $P \to M$. □

43. A glucose solution is administered intravenously into the bloodstream at a constant rate r. As the glucose is added, it is converted into other substances and removed from the bloodstream at a rate that is proportional to the concentration at that time. Thus a model for the concentration $C = C(t)$ of the glucose solution in the bloodstream is

$$\tfrac{dC}{dt} = r - kC,$$

where k is a positive constant.

(a) Suppose that the concentration at time $t = 0$ is C_0. Determine the concentration at any time t by solving the differential equation.

(b) Assuming that $C_0 < \frac{r}{k}$, find $\lim_{t\to\infty} C(t)$ and interpret your answer.

Solution - (a) Rearrange and integrate to get

$$\int \tfrac{1}{r-kC}\, dC = \int dt \implies -\tfrac{1}{k}\ln|r - kC| = t + D,$$

where D is an arbitrary constant. Since $C(0) = C_0$, then

$$-\tfrac{1}{k}\ln|r - kC_0| = 0 + D = D$$

Now

$$\begin{aligned} & -\tfrac{1}{k}\ln|r-kC| = t - \tfrac{1}{k}\ln|r-kC_0| \\ \implies & \ln|r-kC| = -kt + \ln|r-kC_0| \\ \implies & |r-kC| = |r-kC_0|e^{-kt} \\ \implies & r-kC = \pm(r-kC_0)e^{-kt} \\ \implies & C = \tfrac{r}{k} \pm (\tfrac{r}{k} - C_0)e^{-kt} \end{aligned}$$

Since we want $C(0) = C_0$, we conclude that $C(t) = \frac{r}{k} - (\frac{r}{k} - C_0)e^{-kt}$.

(b) As $t \to \infty$, we have $e^{-kt} \to 0$, so $C(t) \to \frac{r}{k}$. In particular, the limiting value of $C(t)$ does not depend on the initial concentration. □

Section 9.4—

9. One model for the spread of a rumor is that the rate of spread is proportional to the product of the fraction y of the population who have heard the rumor and the fraction who have not heard the rumor.

(a) Write a differential equation that is satisfied by y.

(b) Solve the differential equation.

(c) A small town has 1000 inhabitants. At 8 am, 80 people have heard a rumor. By noon, half the town has heard it. At what time will 90% of the population have heard the rumor?

Solution - (a) $\frac{dy}{dt} = ky(1-y)$.

(b) $y(t) = \frac{y_0}{y_0+(1-y_0)e^{-kt}}$, where $y_0 = y(0)$. (For more detail, see pages 608-609 of your textbook.)

(c) Let $t = 0$ correspond to 8 am, and let t be measured in hours. Using the notation of parts (a) and (b), we have $y_0 = \frac{80}{1000} = 0.08$. Now

$$y(t) = \frac{0.08}{0.08+0.92e^{-kt}}$$

We need to determine k. Using the noon condition, we have $y(4) = \frac{1}{2}$. Now

$$\frac{1}{2} = \frac{0.08}{0.08+0.92e^{-4k}} \implies 0.08 + 0.92e^{-4k} = 2 \cdot 0.08 \implies k = -\frac{1}{4}\ln(\frac{0.08}{0.92})$$

To answer the stated question, set $y(t) = 0.9$. Now

$$0.9 = \frac{0.08}{0.08+0.92e^{-kt}} \implies t = -\frac{1}{k}\ln(\frac{\frac{0.08}{0.9} - 0.08}{0.92}) \approx 7.60 \text{ hours} = 7 \text{ hours, } 36 \text{ minutes}$$

At 3:36 pm, 90% of the population will have heard the rumor. □

16. Let c be a positive number. A differential equation of the form

$$\frac{dy}{dt} = ky^{1+c},$$

where k is a positive constant, is called a *doomsday equation* because the exponent in the expression ky^{1+c} is larger than the exponent 1 for natural growth.

(a) Determine the solution that satisfies the initial condition $y(0) = y_0$.

(b) Show that there is a finite time $t = T$ (doomsday) such that $\lim_{t \to T^-} y(t) = \infty$.

(c) An especially prolific breed of rabbits has the growth term $ky^{1.01}$. If 2 such rabbits breed initially and the warren has 16 rabbits after three months, then when is doomsday?

Solution - (a) Rearrange and solve.

$$\int y^{-(1+c)}\, dy = \int k\, dt \implies -\frac{1}{c}y^{-c} = kt+C \implies y^{-c} = -ckt+C \implies y^{c} = \frac{1}{-ckt+C} \implies y = (\frac{1}{-ckt+C})^{1/c}$$

From $y(0) = y_0$, we have $y_0 = (\frac{1}{0+C})^{1/c} = C^{-1/c}$, so $C = y_0^{-c}$. Thus, the solution is

$$y(t) = (\frac{1}{-ckt+y_0^{-c}})^{1/c}$$

(b) To find the time T where the population blows up, set the denominator of $\frac{1}{-ckT+y_0^{-c}}$ equal to zero. Write

$$-ckT + y_0^{-c} = 0 \implies ckT = y_0^{-c} \implies T = \frac{y_0^{-c}}{ck}$$

This is doomsday.

(c) Here $c = 0.01$, $y_0 = 2$, so from part (a) we have $y(t) = (\frac{1}{-0.01kt+2^{-0.01}})^{1/0.01} = (\frac{1}{-0.01kt+2^{-0.01}})^{100}$. To find doomsday, we must first determine k. The warren has 16 rabbits after three months, so

$$16 = y(3) = (\frac{1}{-0.03k+2^{-0.01}})^{100}$$

Solving for k, we have

$$\frac{1}{-0.03k+2^{-0.01}} = 16^{0.01} \implies -0.03k + 2^{-0.01} = 16^{-0.01} \implies k = \frac{2^{-0.01}-16^{-0.01}}{0.03}$$

Now doomsday is given by the answer from part (b),

$$T = \frac{y_0^{-c}}{ck} = \frac{2^{-0.01}}{0.01 \cdot \frac{2^{-0.01}-16^{-0.01}}{0.03}} \approx 146$$

Doomsday occurs after 146 months. □