21-122 - Week 7, Recitation 1

Section 9.2

11. Sketch the direction field of $y' = y - 2x$. Then use it to sketch a solution curve passing through the point $(1, 0)$.

Solution - Check your answer using a slope field generator/drawer from Google. For example, I found one at

http://www.mathscoop.com/calculus/differential-equations/slope-field-generator.php

(Select "Graph a particular solution" and enter $-e^x + 2x + 2$.) □

19. (a) Use Euler's method to estimate the value of $y(0.4)$, where y is the solution of the initial-value problem $y' = y$, $y(0) = 1$.

(i) $h = 0.4$

(ii) $h = 0.2$

(iii) $h = 0.1$

(b) The exact solution of the IVP from (a) is $y = e^x$. Sketch graphs of $y = e^x$ and your approximations from part (a). Use sketches to determine whether your estimates from (a) are overestimates or underestimates.

Solution - (a) In this case, the formula for Euler's Method is $y_n = y_{n-1} + hy_{n-1} = (1+h)y_{n-1}$, $(x_0, y_0) = (0, 1)$.

For $h = 0.4$, we have $y_1 = (1+h)y_0 = (1 + 0.4) \cdot 1 = 1.4$, so $y(0.4) \approx 1.4$.

For $h = 0.2$, write

$$y_1 = (1+h)y_0 = 1.2 \cdot 1 = 1.2, \qquad y_2 = (1+h)y_1 = 1.2 \cdot 1.2 = 1.44$$

For $h = 0.1$, write

$$\begin{aligned} y_1 &= (1+h)y_0 = 1.1 \cdot 1 = 1.1 \\ y_2 &= (1+h)y_1 = 1.1 \cdot 1.1 = 1.1^2 \\ y_3 &= (1+h)y_2 = 1.1 \cdot 1.1^2 = 1.1^3 \\ y_4 &= (1+h)y_3 = 1.1 \cdot 1.1^3 = 1.1^4 \end{aligned}$$

Thus, $y(0.4) \approx 1.1^4 = 1.4641$.

(b) If you sketch $y = e^x$, you should notice that tangent lines to this curve always lie under the curve (this is because $y = e^x$ is concave up). Therefore, the approximation from part (a) is an underestimate. □

Section 9.3

7. Solve $\frac{dy}{dt} = \frac{t}{ye^{y+t^2}}$.

Solution - Rewrite as $\frac{dy}{dt} = \frac{t}{e^{t^2}} \frac{1}{ye^y}$. Now

$$\int ye^y \, dy = \int te^{-t^2} \, dt \implies e^y(y-1) = -\tfrac{1}{2}e^{-t^2} + C$$

(For the first integral, use IBP. For the second, substitute $u = t^2$.) □

13. Solve the initial-value problem $\frac{du}{dt} = \frac{2t+\sec^2 t}{2u}$, $u(0) = -5$.
Solution - Rearrange and integrate to get

$$\int 2u \, du = \int (2t + \sec^2 t) \, dt \implies u^2 = t^2 + \tan t + C \quad (*)$$

To solve for C, substitute $u(0) = -5$ into $(*)$ to get $25 = 0 + 0 + C = C$. Now $u^2 = t^2 + \tan t + 25$ so $u = \pm\sqrt{t^2 + \tan t + 25}$. Since $u(0)$ is negative, then $u = -\sqrt{t^2 + \tan t + 25}$. □

19. Find an equation of the curve that passes through $(0, 1)$ and whose slope at (x, y) is xy.
Solution - This question is asking us to solve the initial-value problem $\frac{dy}{dx} = xy$, $y(0) = 1$. To solve the DE, write

$$\int \tfrac{1}{y} \, dy = \int x \, dy \implies \ln|y| = \tfrac{x^2}{2} + C$$

To solve for C, use $y(0) = 1$. We have

$$\ln|1| = \tfrac{0^2}{2} + C \implies C = \ln|1| = 0$$

Thus, $\ln|y| = \frac{x^2}{2}$. Solving for y, we have

$$|y| = e^{\frac{x^2}{2}} \implies y = e^{\frac{x^2}{2}},$$

where the last step follows from the fact that $y(0)$ is positive. □

21. Solve the differential equation $y' = x + y$ by making the change of variable $u = x + y$.
Solution - Write $u = x + y$, so then $\frac{du}{dx} = 1 + \frac{dy}{dx}$, or $y' = u' - 1$. Now

$$y' = x + y \implies u' - 1 = u \implies u' = 1 + u \quad (*)$$

Assuming $u \neq -1$, write

$$\int \tfrac{1}{1+u} \, du = \int \, dx \implies \ln|u+1| = x + C \implies \ln|y + x + 1| = x + C$$

Rearranging, we have

$$|y + x + 1| = e^C e^x \implies y + x + 1 = \pm e^C e^x \implies y = \pm e^C e^x - x - 1$$

On the other hand, it's not hard to see from $(*)$ that $u = -1$ is a solution of the DE. This corresponds to $y + x = -1$, or $y = -x - 1$. Thus, the general solution of the DE is

$$y = Ke^x - x - 1, \quad K \in \mathbb{R}$$

□