21-122 - Week 6, Recitation 2

Agenda

- Review: 8.1, 8.2 - Arc Length and Area of Surface of Revolution
- 8.1: 7, 12, 33
- 8.2: 5, 11, 15, 25
- (Optional: 8.1 # 32(a),(b))

Review

- Section 8.1: Idea of arclength is to approximate curve by small line segments (see textbook Figure 3, p. 538).
- Arc length of $y = f(x)$, $a \leq x \leq b$ is $L = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2}\, dx$.
- Section 8.2: Idea of surface area is to approximate surface of revolution by "bands" (see textbook Figure 4, p.546).
- Rotation around x-axis: $S = \int 2\pi y\, ds = \int_a^b 2\pi y \sqrt{1 + (\frac{dy}{dx})^2}\, dx = \int_c^d 2\pi y \sqrt{1 + (\frac{dx}{dy})^2}\, dy$
- Rotation around y-axis: $S = \int 2\pi x\, ds = \int_a^b 2\pi x \sqrt{1 + (\frac{dy}{dx})^2}\, dx = \int_c^d 2\pi x \sqrt{1 + (\frac{dx}{dy})^2}\, dy$
- (Choose formula based on parameterization of curve, $y = f(x)$, $a \leq x \leq b$ or $x = g(y)$, $c \leq y \leq d$).

Section 8.1

7. Find the exact length of the curve $y = 1 + 6x^{3/2}$, $0 \leq x \leq 1$.
Solution - We have $\frac{dy}{dx} = 9x^{1/2}$, so now

$$L = \int_0^1 \sqrt{1 + (\tfrac{dy}{dx})^2}\, dx = \int_0^1 \sqrt{1 + 81x}\, dx = \tfrac{1}{81}\tfrac{2}{3}(1 + 81x)^{3/2}\Big|_0^1 = \tfrac{2}{243}(82^{3/2} - 1)$$

□

12. Find the exact length of the curve $y = \ln(\cos x)$, $0 \leq x \leq \frac{\pi}{3}$.
Solution - We have $\frac{dy}{dx} = -\tan x$, so

$$L = \int_0^{\pi/3} \sqrt{1 + (\tfrac{dy}{dx})^2}\, dx = \int_0^{\pi/3} \sqrt{1 + \tan^2 x}\, dx = \int_0^{\pi/3} \sec x\, dx = \ln|\sec x + \tan x|\Big|_0^{\pi/3} = \ln(2 + \sqrt{3})$$

□

33. Find the arc length function for the curve $y = 2x^{3/2}$ with starting point $P_0(1, 2)$.

Solution Write $\frac{dy}{dx} = 3x^{1/2}$, so now $\sqrt{1 + (\frac{dy}{dx})^2} = \sqrt{1+9x}$. Now the arclength function is

$$s(x) = \int_1^x \sqrt{1+9t}\,dt = \tfrac{1}{9}\tfrac{2}{3}(1+9t)^{3/2}\Big|_1^x = \tfrac{2}{27}((1+9x)^{3/2} - 10^{3/2})$$

□

Section 8.2

5, 11. Find the exact area of the surface obtained by rotating the curve about the x-axis.

$$(5)\ y = x^3,\ 0 \le x \le 2, \qquad (11)\ x = \tfrac{1}{3}(y^2+2)^{3/2},\ 1 \le y \le 2$$

Solution - For (5), write the integral down and substitute $u = 1 + 9x^4$, $du = 36x^3\,dx$, so we get

$$S = \int_0^2 2\pi y\sqrt{1+(\tfrac{dy}{dx})^2}\,dx = \int_0^2 2\pi x^3\sqrt{1+9x^4}\,dx = \tfrac{2\pi}{36}\int_1^{145}\sqrt{u}\,du = \tfrac{\pi}{18}\cdot\tfrac{2}{3}u^{3/2}\Big|_1^{145} = \tfrac{\pi}{27}(145^{3/2}-1)$$

For (11), we use the formula $S = \int 2\pi y\sqrt{1+(\frac{dx}{dy})^2}\,dy$. Note that $\frac{dx}{dy} = y\sqrt{y^2+2}$. Write

$$\begin{aligned}
S &= 2\pi\int_1^2 y\sqrt{1+y^2(y^2+2)}\,dy \\
&= 2\pi\int_1^2 y\sqrt{y^4+2y^2+1}\,dy \\
&= 2\pi\int_1^2 y\sqrt{(y^2+1)^2}\,dy \\
&= 2\pi\int_1^2 y(y^2+1)\,dy \\
&= 2\pi(\tfrac{1}{4}y^4 + \tfrac{1}{2}y^2)\Big|_1^2 \\
&= 2\pi(4 + 2 - \tfrac{1}{4} - \tfrac{1}{2}) \\
&= \tfrac{21\pi}{2}
\end{aligned}$$

□

15. The given curve is rotated about the y-axis. Find the area of the resulting surface.

$$x = \sqrt{a^2 - y^2},\ 0 \le y \le \tfrac{a}{2}$$

Solution - Use the formula $S = \int 2\pi x\sqrt{1+(\frac{dx}{dy})^2}\,dy$. We have $\frac{dx}{dy} = -\frac{y}{\sqrt{a^2-y^2}}$. Now

$$S = 2\pi\int_0^{a/2}\sqrt{a^2-y^2}\sqrt{1+\tfrac{y^2}{a^2-y^2}}\,dy = 2\pi\int_0^{a/2}\sqrt{a^2-y^2+y^2}\,dy = 2\pi\int_0^{a/2} a\,dy = 2\pi a\tfrac{a}{2} = \pi a^2$$

□

25. (Gabriel's Horn) Rotate the region $\mathcal{R} = \{(x,y) \mid x \ge 1,\ 0 \le y \le \frac{1}{x}\}$ and we end up with a solid whose volume is finite. Show that the surface area is infinite.

Solution - Use the formula $S = \int 2\pi y\sqrt{1+(\frac{dy}{dx})^2}\,dx$, where $y = \frac{1}{x}$. Now

$$S = 2\pi\int_1^\infty \tfrac{1}{x}\sqrt{1+\tfrac{1}{x^4}}\,dx = 2\pi\int_1^\infty \tfrac{1}{x}\sqrt{\tfrac{1+x^4}{x^4}}\,dx = 2\pi\int_1^\infty \tfrac{\sqrt{1+x^4}}{x^3}\,dx \quad (*)$$

Note that $\frac{\sqrt{1+x^4}}{x^3} \ge \frac{\sqrt{x^4}}{x^3} = \frac{x^2}{x^3} = \frac{1}{x}$ for $x \ge 1$. Since $\int_1^\infty \frac{1}{x}\,dx$ diverges, then so does $(*)$ above. Thus, the surface area is infinite. □