# 21-122 - Week 6, Recitation 1

Agenda

- 7.8: Example 7
- 7.8: #21, 29, 49, 51, 52, 53

**Section 7.8 - Improper Integrals**

Example 7: Evaluate $\int_0^3 \frac{dx}{x-1}$ if possible.
Solution - Wrong approach: Write

$$\int_0^3 \frac{dx}{x-1} = \ln|x-1|\Big|_0^3 = \ln(2) - \ln(1) = \ln(2)$$

This doesn't work because there is an asymptote at $x = 1$ (the integrand is not even defined there!). To approach this as an improper integral, write

$$\int_0^3 \frac{dx}{x-1} = \int_0^1 \frac{dx}{x-1} + \int_1^3 \frac{dx}{x-1}$$

Now

$$\int_0^1 \frac{dx}{x-1} = \ln|x-1|\Big|_0^1 = \lim_{t\to 1^-} (\ln|t-1| - \ln(1)) = -\infty$$

Since $\int_0^1 \frac{dx}{x-1}$ is divergent, then so is $\int_0^3 \frac{dx}{x-1}$. □

21, 29. Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

$$(21) \quad \int_1^\infty \frac{\ln x}{x}\, dx, \qquad (29) \quad \int_{-2}^{14} \frac{dx}{\sqrt[4]{x+2}}$$

Solution - For (21), substitute $u = \ln x$, $du = \frac{dx}{x}$ and write

$$\int_1^\infty \frac{\ln x}{x}\, dx = \int_0^\infty u\, du = \infty$$

Therefore, this integral is divergent. Another way to do this is to split up the integral by writing $\int_1^\infty \frac{\ln x}{x}\, dx = \int_1^e \frac{\ln x}{x}\, dx + \int_e^\infty \frac{\ln x}{x}\, dx$. Then note that $\frac{\ln x}{x} \geq \frac{1}{x}$ for $x \geq e$, so by the Comparison Test, $\int_e^\infty \frac{\ln x}{x}\, dx$ is divergent, so then $\int_1^\infty \frac{\ln x}{x}\, dx$ is divergent.

For (29), write

$$\int_{-2}^{14} \frac{dx}{\sqrt[4]{x+2}} = \int_{-2}^{14} (x+2)^{-1/4}\, dx = \frac{4}{3}(x+2)^{3/4}\Big|_{-2}^{14} = \frac{4}{3}(16^{3/4} - 0) = \frac{32}{3},$$

so this integral is convergent. □

49, 51, 52, 53. Use the Comparison Test to determine whether the intergal is convergent or divergent.

$$(49)\ \int_0^\infty \frac{x}{x^3+1}\,dx, \qquad (51)\ \int_1^\infty \frac{x+1}{\sqrt{x^4-x}}\,dx, \qquad (52)\ \int_0^\infty \frac{\arctan x}{2+e^x}\,dx, \qquad (53)\ \int_0^1 \frac{\sec^2 x}{x\sqrt{x}}\,dx$$

Solution - For (49), write $\int_0^\infty \frac{x}{x^3+1}\,dx = \int_0^1 \frac{x}{x^3+1}\,dx + \int_1^\infty \frac{x}{x^3+1}\,dx$. Now for $x \geq 1$, we have $\frac{x}{x^3+1} \leq \frac{x}{x^3} = \frac{1}{x^2}$. Since $\int_1^\infty \frac{1}{x^2}\,dx$ is convergent, then so is $\int_1^\infty \frac{x}{x^3+1}\,dx$, and so is the original integral.

For (51), observe that $x^4 - x \leq x^4$ for $x \geq 1$, so then

$$\frac{x+1}{\sqrt{x^4-x}} \geq \frac{x+1}{\sqrt{x^4}} = \frac{x+1}{x^2} \geq \frac{x}{x^2} = \frac{1}{x}$$

Since $\int_1^\infty \frac{1}{x}\,dx$ is divergent, then by the Comparison Test, so is the original integral.

For (52), we can write $\frac{\arctan x}{2+e^x} \leq \frac{\pi}{2}\frac{1}{2+e^x} \leq \frac{\pi}{2}\frac{1}{e^x}$. Now

$$\int_0^\infty \frac{\pi}{2}\frac{1}{e^x}\,dx = -\frac{\pi}{2}e^{-x}\Big|_0^\infty = \frac{\pi}{2},$$

so $\int_0^\infty \frac{\pi}{2}\frac{1}{e^x}\,dx$ is convergent. By the Comparison Test, so is the original integral.

For (53), we'd like to compare $\int_0^1 \frac{\sec^2 x}{x\sqrt{x}}\,dx$ to $\int_0^1 \frac{1}{x\sqrt{x}}\,dx$ or something like that. To do this, note that on $[0,1]$, we have $\sec^2 x \geq 1$ (since $\cos^2 x \leq 1$). Thus, $\frac{\sec^2 x}{x\sqrt{x}} \geq \frac{1}{x\sqrt{x}}$. Now

$$\int_0^1 \frac{1}{x\sqrt{x}}\,dx = \int_0^1 x^{-3/2}\,dx = -2x^{-1/2}\Big|_0^1 = 2(\lim_{t\to 0^+} t^{-1/2} - 1) = \infty$$

By the Comparison Test, since $\int_0^1 \frac{1}{x\sqrt{x}}\,dx$ is divergent, then so is $\int_0^1 \frac{\sec^2 x}{x\sqrt{x}}\,dx$. $\square$