21-122 - Week 4, Recitation 2

Agenda

- Review: Section 7.7 - Approximate Integration
- 7.7: 3, 19
- Quiz 1
- Return HW 3

Review

- Sometimes we can't find the exact value of a definite integral, so numerical integration is useful.
- Midpoint Rule:
$$\int_a^b f(x)\,dx \approx M_n = \Delta x[f(\overline{x}_1) + f(\overline{x}_2) + \cdots + f(\overline{x})_n],$$
where $\Delta x = \frac{b-a}{n}$ and $\overline{x}_i = \frac{1}{2}(x_{i-1} + x_i)$, the midpoint of $[x_{i-1}, x_i]$.
- Trapezoidal Rule:
$$\int_a^b f(x)\,dx \approx T_n = \tfrac{\Delta x}{2}[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)],$$
where $\Delta x = \frac{b-a}{n}$.
- Error bounds: Suppose $|f''(x)| \le K$ for all $a \le x \le b$. If E_T and E_M are the errors in the Trapezoidal and Midpoint Rules, then
$$|E_T| \le \tfrac{K(b-a)^3}{12n^2}, \qquad |E_M| \le \tfrac{K(b-a)^3}{24n^2}$$

Section 7.7

3. Estimate $\int_0^1 \cos(x^2)\,dx$ using (a) the Trapezoidal Rule and (b) the Midpoint Rule, each with $n = 4$. From a graph of the integrand, decide whether your answers are underestimates or overestimates. What can you conclude about the true value of the integral?
Solution - $f(x) = \cos(x^2)$.

Using the Trapezoidal Rule, we have

$$\begin{aligned}\int_0^1 \cos(x^2)\,dx &\approx \tfrac{1/4}{2}[f(0) + 2f(\tfrac{1}{4}) + 2f(\tfrac{1}{2}) + 2f(\tfrac{3}{4}) + f(1)]\\ &= \tfrac{1}{8}[\cos(0) + 2\cos(\tfrac{1}{16}) + 2\cos(\tfrac{1}{4}) + 2\cos(\tfrac{9}{16}) + \cos(1)]\\ &\approx 0.8958\end{aligned}$$

Using the Midpoint Rule, we have

$$\begin{aligned}\int_0^1 \cos(x^2)\,dx &\approx \tfrac{1}{4}(f(\tfrac{1}{8}) + f(\tfrac{3}{8}) + f(\tfrac{5}{8}) + f(\tfrac{7}{8}))\\ &= \tfrac{1}{4}(\cos(\tfrac{1}{64}) + \cos(\tfrac{9}{64}) + \cos(\tfrac{25}{64}) + \cos(\tfrac{49}{64}))\\ &\approx 0.9089\end{aligned}$$

Looking at the graph, it appears that the estimate from part (a) is an underestimate, while the estimate from part (b) is an overestimate. Therefore, the true value of the integral lies somewhere between these estimates. □

19. (a) Find the approximations T_8 and M_8 for the integral $\int \cos(x^2)\, dx$.
(b) Estimate the errors in the approximations of part (a).
(c) How large must we choose n so that the approximations T_n and M_n to the integral in part (a) are accurate to within 0.0001?

Solution - (a)

$$\begin{aligned} T_8 &= \tfrac{1/8}{2}[f(0) + 2f(\tfrac{1}{8}) + 2f(\tfrac{2}{8}) + \cdots + 2f(\tfrac{7}{8}) + f(1)] \\ &= \tfrac{1}{16}[\cos(0) + 2\cos(\tfrac{1}{64}) + 2\cos(\tfrac{2^2}{64}) + 2\cos(\tfrac{3^2}{64}) + \cdots + 2\cos(\tfrac{7^2}{64}) + \cos(1)] \\ &\approx 0.9023 \end{aligned}$$

$$\begin{aligned} M_8 &= \tfrac{1}{8}[f(\tfrac{1}{16}) + f(\tfrac{3}{16}) + f(\tfrac{5}{16}) + \cdots + f(\tfrac{15}{16})] \\ &= \tfrac{1}{8}[\cos(\tfrac{1}{16^2}) + \cos(\tfrac{3^2}{16^2}) + \cdots + \cos(\tfrac{15^2}{16^2})] \\ &\approx 0.9056 \end{aligned}$$

(b) To estimate the errors, we need a bound on $|f''(x)|$ on the interval $[0, 1]$. We have $f'(x) = -2x\sin(x^2)$, so

$$f''(x) = -2\sin(x^2) - 4x^2\cos(x^2)$$

On the interval $[0, 1]$, $\sin(x^2)$ and $x^2\cos(x^2)$ are positive, so

$$|f''(x)| = |-2\sin(x^2) - 4x^2\cos(x^2)| = 2\sin(x^2) + 4x^2\cos(x^2) \leq 2 \cdot 1 + 4 \cdot 1 \cdot 1 = 6$$

Thus, the errors for the approximations of part (a) have the following upper bounds.

$$|E_T| \leq \tfrac{6(1-0)^3}{12 \cdot 8^2} \approx 0.0078, \qquad |E_M| \leq \tfrac{6 \cdot (1-0)^3}{24 \cdot 8^2} \approx 0.0039$$

(c) For T_n, we want $|E_T| \leq 0.0001$, so write

$$\tfrac{6 \cdot (1-0)^3}{12n^2} \leq 0.0001 \implies \tfrac{1}{2n^2} \leq 0.0001 \implies \tfrac{1}{2 \cdot 0.0001} \leq n^2 \implies n^2 \geq 5000 \implies n \geq \sqrt{5000} \approx 70.71$$

so choose $n = 71$ for T_n. For M_n, we want $|E_M| \leq 0.0001$, so write

$$\tfrac{6 \cdot (1-0)^3}{24n^2} \leq 0.0001 \implies \tfrac{1}{4n^2} \leq 0.0001 \implies \tfrac{1}{4 \cdot 0.0001} \leq n^2 \implies n^2 \geq 2500 \implies n \geq \sqrt{2500} = 50$$

so choose $n = 50$ for M_n. □