21-122 - Week 4, Recitation 1

Agenda

- Announcements
- Sec 7.5: Strategy for Integration
  - Example 5
  - 17, 23, 41, 49
  - 45, 57, 63 (time permitting)
- Submit HW 3
- Return HW 2

Announcements

- There will be a 20-minute quiz at the end of Thursday's recitation. You must take the quiz in the recitation section you are registered in.
- Don't copy from one another. If students are caught doing this, a grade of zero will be assigned.
- Common mistake: sign errors. A lot of students wrote $\int \frac{1}{4-x}\ dx = \ln|4-x| + C$. This is wrong, the answer is $-\ln|4-x| + C$. This sign error will matter a lot when we do differential equations.
- The integral $\int \frac{1}{(x^2+1)^2}\ dx$. This can be done by trigonometric substitution, $x = \tan\theta$, $dx = \sec^2\theta\ d\theta$. Now

$$\begin{aligned}\int \frac{1}{(x^2+1)^2}\ dx = \int \frac{1}{(\tan^2\theta+1)^2}\sec^2\theta\ d\theta = \int \frac{1}{\sec^4\theta}\sec^2\theta\ d\theta &= \int \cos^2\theta\ d\theta \\ &= \frac{1}{2}\int (1+\cos 2\theta)\ d\theta \\ &= \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C \\ &= \frac{1}{2}(\theta + \sin\theta\cos\theta) + C \\ &= \frac{1}{2}(\arctan x + \frac{x}{x^2+1}) + C\end{aligned}$$

Section 7.5

Methods

- Simplify integrand if possible
- Substitution
- Integration by parts
- Partial fractions
- Trigonometric substitution

Sometimes guesswork is necessary.

Example 5: Evaluate $\int \sqrt{\frac{1-x}{1+x}}\ dx$.
Solution - Manipulate the integrand. Multiply by $\frac{\sqrt{1-x}}{\sqrt{1-x}}$.

$$\int \sqrt{\tfrac{1-x}{1+x}}\ dx = \int \tfrac{1-x}{\sqrt{1-x^2}}\ dx = \int \tfrac{1}{\sqrt{1-x^2}}\ dx - \int \tfrac{x}{\sqrt{1-x^2}}\ dx = \sin^{-1} x + \sqrt{1-x^2} + C$$

(the second integral can be done by substituting $u = \sqrt{1-x^2}$, $du = -\frac{x}{\sqrt{1-x^2}}\ dx$). □

17. Evaluate $\int_0^\pi t\cos^2 t\ dt$.
Solution - First, simplify by writing $\int_0^\pi t\cos^2 t\ dt = \frac{1}{2}\int_0^\pi t(1+\cos 2t)\ dt$. Now use integration by parts, where $u = t$, $dv = (1+\cos 2t)\ dt$, so $du = dt$ and $v = t + \frac{1}{2}\sin 2t$. We have

$$\begin{aligned}\int_0^\pi t\cos^2 t\ dt = \tfrac{1}{2}\int_0^\pi t(1+\cos 2t)\ dt &= \tfrac{1}{2}[t(t+\tfrac{1}{2}\sin 2t)\Big|_0^\pi - \int_0^\pi (t+\tfrac{1}{2}\sin 2t)\ dt] \\ &= \tfrac{1}{2}[\pi^2 - (\tfrac{1}{2}t^2 - \tfrac{1}{4}\cos 2t)\Big|_0^\pi] \\ &= \tfrac{1}{2}[\pi^2 - \tfrac{1}{2}\pi^2] \\ &= \tfrac{1}{4}\pi^2\end{aligned}$$

□

23. Evaluate $\int_0^1 (1+\sqrt{x})^8\ dx$.
Solution - Substitute $u = 1+\sqrt{x}$, so $du = \frac{1}{2\sqrt{x}}\ dx$, or $dx = 2\sqrt{x}\ du = 2(u-1)\ du$. Now

$$\int_0^1 (1+\sqrt{x})^8\ dx = 2\int_1^2 u^8(u-1)\ du = 2\int_1^2 u^9 - u^8\ du = 2[\tfrac{1}{10}u^{10} - \tfrac{1}{9}u^9]\Big|_1^2 = \tfrac{4097}{45}$$

□

41. Evaluate $\int \theta\tan^2\theta\ d\theta$.
Solution - Use $\tan^2\theta = \sec^2\theta - 1$ and then use integration by parts.

$$\begin{aligned}\int \theta\tan^2\theta\ d\theta = \int \theta\sec^2\theta\ d\theta - \int \theta\ d\theta &= [\theta\tan\theta - \int \tan\theta\ d\theta] - \tfrac{1}{2}\theta^2 \\ &= \theta\tan\theta - \ln|\sec\theta| - \tfrac{1}{2}\theta^2 + C\end{aligned}$$

□

49. Evaluate $\int \frac{1}{x\sqrt{4x+1}}\ dx$.
Solution - Substitute $u = \sqrt{4x+1}$, so then $du = \frac{2}{\sqrt{4x+1}}\ dx$, or $dx = \frac{1}{2}\sqrt{4x+1}\ du = \frac{1}{2}u\ du$. Note that $u^2 = 4x+1$, so $x = \frac{1}{4}(u^2-1)$, so $\frac{1}{x} = \frac{4}{u^2-1}$. Now using partial fractions, we have

$$\begin{aligned}\int \tfrac{1}{x\sqrt{4x+1}}\ dx = \int \tfrac{4}{(u^2-1)u}\cdot\tfrac{1}{2}u\ du = \int \tfrac{2}{u^2-1}\ du &= \int (\tfrac{1}{u-1} - \tfrac{1}{u+1})\ du \\ &= \ln|u-1| - \ln|u+1| + C \\ &= \ln|\tfrac{u-1}{u+1}| + C \\ &= \ln|\tfrac{\sqrt{4x+1}-1}{\sqrt{4x+1}+1}| + C\end{aligned}$$

□

45. Evaluate $\int x^5 e^{-x^3}\,dx$.
Solution - First substitute $u = x^3$, $du = 3x^2\,dx$, so $x^2\,dx = \frac{1}{3}\,du$ Now

$$\int x^5 e^{-x^3}\,dx = \int ue^{-u}\cdot\tfrac{1}{3}du = \tfrac{1}{3}\int ue^{-u}\,du$$

Now use IBP to get

$$\int ue^{-u}\,du = -ue^{-u} + \int e^{-u}\,du = -ue^{-u} - e^{-u} + C$$

Thus,

$$\int x^5 e^{-x^3}\,dx = \tfrac{1}{3}(-ue^{-u} - e^{-u}) + C = -\tfrac{1}{3}x^3 e^{-x^3} - \tfrac{1}{3}e^{-x^3} + C = -\tfrac{1}{3}(x^3+1)e^{-x^3} + C$$

□

57. Evaluate $\int x\sqrt[3]{x+c}\,dx$.
63. Evaluate $\int \sqrt{x}e^{\sqrt{x}}\,dx$.