21-122 - Week 3, Recitation 2

Agenda

- Quadratic Polynomials
- Review: Partial Fractions
- Section 7.4: Example 3, Example 4, 7.4: 2, 4, 6, 14, 15, 21
- Return HW 2

Quadratic Polynomials

- To show that a quadratic function $ax^2 + bx + c$ is irreducible (i.e. it can't be factored), use the quadratic formula to show that it has no real roots, i.e. $b^2 - 4ac < 0$.
- Examples: For $x^2 + x + 1$, we have $b^2 - 4ac = 1 - 4 = -3$, so this polynomial is irreducible. Similarly, for $x^2 - x + 1$, we have $b^2 - 4ac = 1 - 4 = -3$, so this polynomial is irreducible.
- When factoring a quadratic $ax^2 + bx + c$, we often guess the factorization. Sometimes it's better to simply find the roots x_1 and x_2 of the quadratic, then

$$ax^2 + bx + c = a(x - x_1)(x - x_2)$$

- Examples:

$$-6x^2 + 15x + 36 = -6(x-4)(x+\tfrac{3}{2}), \qquad 6x^2 + 13x + 6 = 6(x+\tfrac{2}{3})(x+\tfrac{3}{2})$$

$$20x^2 - 17x - 63 = 20(x-\tfrac{9}{4})(x+\tfrac{7}{5})$$

Section 7.4

Example 3: Find $\int \frac{dx}{x^2-a^2}$, where $a \neq 0$.
Solution - Write

$$\frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)} = \frac{A}{x-a} + \frac{B}{x+a}$$

Therefore, $A(x+a) + B(x-a) = 1$. Now set $x = a$ to get $A(2a) = 1$, so $A = \frac{1}{2a}$. Set $x = -a$ to get $B(-2a) = 1$, so $B = -\frac{1}{2a}$. Now

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\int \left(\frac{1}{x-a} - \frac{1}{x+a}\right) dx = \frac{1}{2a}(\ln|x-a| - \ln|x+a|) + C = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

□

Example 4: Find $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx$.
Solution - Since degree(numerator) = 4 and degree(denominator) = 3, we first do long division. We get

$$\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = x + 1 + \frac{4x}{x^3-x^2-x+1}$$

Now factorize $x^3 - x^2 - x + 1$ by writing

$$x^3 - x^2 - x + 1 = x^2(x-1) - (x-1) = (x^2-1)(x-1) = (x-1)^2(x+1)$$

Next, expand

$$\frac{4x}{x^3-x^2-x+1} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

Clearing the denominators, we have

$$4x = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$

Setting $x = 1$, we have $4 = 2B$, so $B = 2$. Setting $x = -1$, we have $-4 = C(-2)^2 = 4C$, so $C = -1$. There's no nice way to get A directly, so let's just use the easy value $x = 0$ and see what we get. Setting $x = 0$, we have

$$0 = A(-1)(1) + 2(1) - 1(-1)^2 = -A + 2 - 1 \implies A = 2 - 1 = 1$$

Now

$$\begin{aligned}\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx &= \int (x+1+\frac{4x}{x^3-x^2-x+1}\,dx \\ &= \int (x+1+\frac{1}{x-1}+\frac{2}{(x-1)^2}-\frac{1}{x+1})\,dx \\ &= \frac{1}{2}x^2 + x + \ln|x-1| - \frac{2}{x-1} - \ln|x+1| + C_1 \\ &= \frac{1}{2}x^2 + x + \ln|\frac{x-1}{x+1}| - \frac{2}{x-1} + C_1\end{aligned}$$

where C_1 is a constant. □

Recall - $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan(\frac{x}{a}) + C$. We will need this later today.

(2,4,6) Write out the form of the partial fraction decomposition of the function. Do not determine the numerical values of the coefficients.

$2(a)$	$\frac{x}{x^2+x-2}$	(b)	$\frac{x^2}{x^2+x-2}$
$4(a)$	$\frac{x^4-2x^3+x^2+2x-1}{x^2-2x+1}$	(b)	$\frac{x^2-1}{x^3+x^2+x}$
$6(a)$	$\frac{t^6+1}{t^6+t^3}$	(b)	$\frac{x^5+1}{(x^2-x)(x^4+2x^2+1)}$

<u>Solutions</u> - In (2)(a) and (2)(b), the denominator factors as $x^2+x-2 = (x+2)(x-1)$. Then

$$\frac{x}{x^2+x-2} = \frac{A}{x+2} + \frac{B}{x-1}$$

For (b), we have to perform long division first. We get

$$\frac{x^2}{x^2+x-2} = 1 + \frac{-x+2}{x^2+x-2} = 1 + \frac{A}{x+2} + \frac{B}{x-1}$$

For 4(a), we perform long divison and factorize $x^2-2x+1 = (x-1)^2$. We get.

$$\frac{x^4-2x^3+x^2+2x-1}{x^2-2x+1} = x^2 + \frac{2x-1}{(x-1)^2} = x^2 + \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

For (b), factorize $x^3+x^2+x = x(x^2+x+1)$ and note that the quadratic is irreducible. Therefore,

$$\frac{x^2-1}{x^3+x^2+x} = \frac{A}{x} + \frac{Bx+C}{x^2+x+1}$$

For 6(a), perform long division and factorize $t^6+t^3 = t^3(t^3+1) = t^3(t+1)(t^2-t+1)$. The quadratic is irreducible, so we get

$$\frac{t^6+1}{t^6+t^3} = 1 + \frac{-t^3+1}{t^6+t^3} = 1 + \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t^3} + \frac{D}{t+1} + \frac{Ex+F}{t^2-t+1}$$

Finally, for (b) we factorize $(x^2 - x)(x^4 + 2x^2 + 1) = x(x-1)(x^2+1)^2$. Since $x^2 + 1$ is irreducible, we have

$$\frac{x^5+1}{(x^2-x)(x^4+2x^2+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2}$$

□

(14) Evaluate $\int \frac{1}{(x+a)(x+b)}\, dx$.
<u>Solution</u> - If $a = b$, this is easy, we have $\int \frac{1}{(x+a)^2}\, dx = -\frac{1}{x+a} + C$. If $a \neq b$, write

$$\frac{1}{(x+a)(x+b)} = \frac{A}{x+a} + \frac{B}{x+b}$$

Clearing denominators, we have $1 = A(x+b) + B(x+a)$. Setting $x = -a$, we have $1 = A(b-a)$, so $A = \frac{1}{b-a}$. Setting $x = -b$, we have $1 = B(a-b) = -B(b-a)$, so $B = -\frac{1}{b-a}$. Now

$$\int \frac{1}{(x+a)(x+b)}\, dx = \frac{1}{b-a}\int (\frac{1}{x+a} - \frac{1}{x+b})\, dx = \frac{1}{b-a}(\ln|x+a| - \ln|x+b|) + C = \frac{1}{b-a}\ln|\frac{x+a}{x+b}| + C$$

□

(15) Evaluate $\int_3^4 \frac{x^3-2x^2-4}{x^3-2x^2}\, dx$.
<u>Solution</u> - Perform long division (or just do this by inspection) to get $\frac{x^3-2x^2-4}{x^3-2x^2} = 1 - \frac{4}{x^3-2x^2}$. Next, write

$$-\frac{4}{x^3-2x^2} = -\frac{4}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

Clearing denominators, we have

$$-4 = Ax(x-2) + B(x-2) + Cx^2$$

Setting $x = 2$, we have $4C = -4$, so $C = -1$. Setting $x = 0$, we have $-4 = -2B$, so $B = 2$. To get A, set $x = 1$, so

$$-4 = -A - B + C \implies A = -B + C + 4 = -2 - 1 + 4 = 1$$

Now

$$\begin{aligned}
\int_3^4 \frac{x^3-2x^2-4}{x^3-2x^2}\, dx &= \int_3^4 (1 - \frac{4}{x^3-2x^2})\, dx \\
&= \int_3^4 (1 + \frac{1}{x} + \frac{2}{x^2} - \frac{1}{x-2})\, dx \\
&= [x + \ln|x| - \frac{2}{x} - \ln|x-2|]\Big|_3^4 \\
&= (4-3) + (\ln(4) - \ln(3)) + (\frac{2}{3} - \frac{2}{4}) + (\ln(1) - \ln(2)) \\
&= 1 + \ln\frac{4}{3} + \frac{1}{6} - \ln(2) \\
&= \frac{7}{6} + \ln\frac{2}{3}
\end{aligned}$$

□

(21) Evalute $\int \frac{x^3+4}{x^2+4}\, dx$.
<u>Solution</u> - Perform long division to get $\frac{x^3+4}{x^2+4} = x + \frac{4-4x}{x^2+4}$. Since x^2+4 is irreducible, this is already in partial fraction form, so we have

$$\int \frac{x^3+4}{x^2+4}\, dx = \int (x + \frac{4-4x}{x^2+4})\, dx = \int (x + 4\frac{1}{x^2+4} - 4\frac{x}{x^2+4})\, dx = \frac{1}{2}x^2 + 2\arctan(\frac{x}{2}) - 2\ln(x^2+4) + C$$