21-122 - Week 3, Recitation 1

Agenda

- Integral of $\csc(x)$
- Review
- Section 7.3 (Trigonometric Substitution): 23, 26, 29, 30
- Section 7.4 (Partial Fractions): Example 1
- Polynomial long division

Integral of Cosecant

I'll demonstrate two ways that we can calculate this integral. The first way uses a trick that would be hard to discover on your own. The second way

For the first trick, write

$$\int \csc x \, dx = \int \csc x \tfrac{\csc x - \cot x}{\csc x - \cot x} \, dx = \int \tfrac{\csc^2 x - \csc x \cot x}{\csc x - \cot x} \, dx$$

Substitute $u = \csc x - \cot x$, so $du = (-\csc x \cot x + \csc^2 x) \, dx$. Now

$$\int \csc x \, dx = \int \tfrac{du}{u} = \ln|u| + C = \ln|\csc x - \cot x| + C$$

Another way to calculate this integral is to write $\csc x = \sec(\frac{\pi}{2} - x)$ and substitute $u = \frac{\pi}{2} - x$, so $du = -dx$. Now

$$\int \csc x = \int \sec(\tfrac{\pi}{2} - x) \, dx = -\int \sec u \, du = -\ln|\sec u + \tan u| + C = -\ln|\csc x + \cot x| + C,$$

This is an equivalent expression to the previous answer, because

$$-\ln|\csc x + \cot x| = \ln|\tfrac{1}{\csc x + \cot x}| = \ln|\tfrac{\csc x - \cot x}{\csc^2 x - \cot^2 x}| = \ln|\csc x - \cot x|$$

Review

- To evaluate an integral involving $\sqrt{a^2 - x^2}$, substitute $x = a \sin\theta$ and use $1 - \sin^2\theta = \cos^2\theta$.
- To evaluate an integral involving $\sqrt{a^2 + x^2}$, substitute $x = a \tan\theta$ and use $1 + \tan^2\theta = \sec^2\theta$.
- To evaluate an integral involving $\sqrt{x^2 - a^2}$, substitute $x = a \sec\theta$ and use $\sec^2\theta - 1 = \tan^2\theta$.
- **Note:** These substitutions may also be useful even if no square root appears in the question. Keep this in mind.

Section 7.3

(23) Evaluate $\int \sqrt{5 + 4x - x^2} \, dx$.
Solution - First complete the square by writing

$$5 + 4x - x^2 = -(x^2 - 4x - 5) = -(x^2 - 4x + 4 - 9) = -(x-2)^2 + 9$$

Performing the substitution $u = x-2$, $du = dx$, we have $\int \sqrt{5+4x-x^2}\,dx = \int \sqrt{9-u^2}\,du$. Now substitute $u = 3\sin\theta$, $du = 3\cos\theta\,d\theta$ to get

$$\begin{aligned}\int \sqrt{5+4x-x^2}\,dx = \int \sqrt{9-9\sin^2\theta}\,3\cos\theta\,d\theta &= \int 9\sqrt{\cos^2\theta}\cos\theta\,d\theta \\ &= 9\int \cos^2\theta\,d\theta \\ &= \tfrac{9}{2}\int (1+\cos 2\theta)\,d\theta \\ &= \tfrac{9}{2}\theta + \tfrac{9}{4}\sin 2\theta + C \\ &= \tfrac{9}{2}\sin^{-1}(\tfrac{x-2}{3}) + \tfrac{9}{2}\sin\theta\cos\theta + C \\ &= \tfrac{9}{2}\sin^{-1}(\tfrac{x-2}{3}) + \tfrac{9}{2}\tfrac{u}{3}\cdot\tfrac{\sqrt{9-u^2}}{3} + C \\ &= \tfrac{9}{2}\sin^{-1}(\tfrac{x-2}{3}) + \tfrac{1}{2}u\sqrt{9-u^2} + C\end{aligned}$$

Note, $\theta = \sin^{-1}(\frac{u}{3}) = \sin^{-1}(\frac{x-2}{3})$ and

$$\tfrac{9}{4}\sin 2\theta = \tfrac{9}{2}\sin\theta\cos\theta = \tfrac{9}{2}\cdot\tfrac{u}{3}\cdot\tfrac{\sqrt{9-u^2}}{3} = \tfrac{1}{2}u\sqrt{9-u^2} = \tfrac{1}{2}(x-2)\sqrt{9-(x-2)^2}$$

Thus,

$$\int \sqrt{5+4x-x^2}\,dx = \tfrac{9}{2}\sin^{-1}(\tfrac{x-2}{3}) + \tfrac{1}{2}(x-2)\sqrt{9-(x-2)^2} + C$$

□

(26) Evaluate $\int \frac{x^2}{(3+4x-4x^2)^{3/2}}\,dx$.
Solution - We begin by completing the square of the denominator. Write

$$3+4x-4x^2 = -4(x^2-x-\tfrac{3}{4}) = -4(x^2-x+\tfrac{1}{4}-1) = -4(x-\tfrac{1}{2})^2+4$$

Substituting $u = x-\frac{1}{2}$, $du = dx$, we have

$$\begin{aligned}\int \tfrac{x^2}{(3+4x-4x^2)^{3/2}}\,dx &= \int \tfrac{x^2}{(4-4(x-\frac{1}{2})^2)^{3/2}}\,dx \\ &= \int \tfrac{x^2}{8(1-(x-\frac{1}{2})^2)^{3/2}}\,dx \\ &= \tfrac{1}{8}\int \tfrac{(u+\frac{1}{2})^2}{(1-u^2)^{3/2}}\,du \\ &= \tfrac{1}{8}\int \tfrac{(\sin\theta+\frac{1}{2})^2}{(1-\sin^2\theta)^{3/2}}\cos\theta\,d\theta \qquad (u=\sin\theta) \\ &= \tfrac{1}{8}\int \tfrac{\sin^2\theta+\sin\theta+\frac{1}{4}}{\cos^2\theta}\,d\theta \\ &= \tfrac{1}{8}\int (\tan^2\theta+\tan\theta\sec\theta+\tfrac{1}{4}\sec^2\theta)\,d\theta \\ &= \tfrac{1}{8}\int (\tfrac{5}{4}\sec^2\theta+\tan\theta\sec\theta-1)\,d\theta \\ &= \tfrac{1}{8}(\tfrac{5}{4}\tan\theta+\sec\theta-\theta)+C \\ &= \tfrac{1}{8}(\tfrac{5}{4}\tfrac{u}{\sqrt{1-u^2}}+\tfrac{1}{\sqrt{1-u^2}}-\sin^{-1}(u))+C \\ &= \tfrac{1}{8}(\tfrac{5}{4}\tfrac{x-1/2}{\sqrt{1-(x-\frac{1}{2})^2}}+\tfrac{1}{\sqrt{1-(x-\frac{1}{2})^2}}-\sin^{-1}(x-\tfrac{1}{2}))+C\end{aligned}$$

□

(29) Evaluate $\int x\sqrt{1-x^4}\,dx$.
Solution - Write

$$\begin{aligned}\int x\sqrt{1-x^4}\,dx &= \tfrac{1}{2}\int \sqrt{1-u^2}\,du \qquad (u=x^2,\ du=2x\,dx)\\ &= \tfrac{1}{2}\int \sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta \qquad (u=\sin\theta)\\ &= \tfrac{1}{2}\int \cos^2\theta\,d\theta\\ &= \tfrac{1}{4}\int (1+\cos 2\theta)\,d\theta\\ &= \tfrac{1}{4}(\theta+\tfrac{1}{2}\sin 2\theta)+C\\ &= \tfrac{1}{4}\sin^{-1}(x^2)+\tfrac{1}{8}\sin 2\theta + C\end{aligned}$$

Since $\frac{1}{8}\sin 2\theta = \frac{1}{4}\sin\theta\cos\theta = \frac{1}{4}x^2\sqrt{1-x^2}$, then

$$\int x\sqrt{1-x^4}\,dx = \tfrac{1}{4}\sin^{-1}(x^2)+\tfrac{1}{4}x^2\sqrt{1-x^4}+C$$

□

(30) Evaluate $\int_0^{\pi/2} \frac{\cos t}{\sqrt{1+\sin^2 t}}\,dt$.
Solution - Write

$$\begin{aligned}\int_0^{\pi/2} \frac{\cos t}{\sqrt{1+\sin^2 t}}\,dt &= \int_0^1 \frac{1}{\sqrt{1+u^2}}\,du \qquad (u=\sin t)\\ &= \int_0^{\pi/4} \frac{\sec^2\theta}{\sqrt{1+\tan^2\theta}}\,d\theta \qquad (u=\tan\theta)\\ &= \int_0^{\pi/4} \sec\theta\,d\theta\\ &= \ln|\sec\theta+\tan\theta|\Big|_0^{\pi/4}\\ &= \ln|\sqrt{2}+1| - \ln|1|\\ &= \ln(1+\sqrt{2})\end{aligned}$$

□

Section 7.4

Example 1: Find $\int \frac{x^3+x}{x-1}\,dx$.
Solution - Performing polynomial long division, we have

$$\frac{x^3+x}{x-1} = x^2+x+2+\frac{2}{x-1}$$

Thus,

$$\int \frac{x^3+x}{x-1}\,dx = \int (x^2+x+2+\tfrac{2}{x-1})\,dx = \tfrac{1}{3}x^3+\tfrac{1}{2}x^2+2x+2\ln|x-1|+C$$

□

Polynomial Long Division: Examples

1. Divide $3x^3-5x^2+10x-3$ by $3x+1$. (Answer: $x^2-2x+4-\frac{7}{3x+1}$)
2. Divide $4x^3-2x^2-3$ by $2x^2-1$. (Answer: $2x-1+\frac{2x-4}{2x^2-1}$)
3. Divide $3x^3+4x+11$ by x^2-3x+2. (Answer: $3x+9+\frac{25x-7}{x^2-3x+2}$)