21-122 - Integration and Approximation
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21-122 - Week 15, Recitation 2

Section 10.3

61. Find the points on the curve $r = 3\cos\theta$ where the tangent line is horizontal or vertical.

Solution - Since $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$, then horizontal tangents correspond to $\frac{dy}{d\theta} = 0$ (assuming $\frac{dx}{d\theta} \neq 0$). Vertical tangents correspond to $\frac{dx}{d\theta} = 0$ (assuming $\frac{dy}{d\theta} \neq 0$).

- **Horizontal tangents**: $y = r\sin\theta = 3\cos\theta\sin\theta$, so $\frac{dy}{d\theta} = 3\cos^2\theta - 3\sin^2\theta$. Setting this equal to zero, we have $\cos^2\theta - \sin^2\theta = 0$, or $\cos 2\theta = 0$. This gives rise to $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$. Finding the r-values corresponding to these θ-values, we end up with the two points

$$(\tfrac{3\sqrt{2}}{2}, \tfrac{\pi}{4}), \qquad (\tfrac{3\sqrt{2}}{,} \tfrac{7\pi}{4})$$

  The tangents are indeed horizontal here, because $\frac{dx}{d\theta} \neq 0$ at these points.

- **Vertical tangents**: $x = r\cos\theta = 3\cos^2\theta$, so $\frac{dx}{d\theta} = -6\cos\theta\sin\theta$. Setting this equal to zero, we have $-3\sin 2\theta = 0$, or $\sin 2\theta = 0$. We have $\theta = 0$, $\theta = \frac{\pi}{2}$, $\theta = \pi$, and $\theta = \frac{3\pi}{2}$. Finding the r-values corresponding to these θ-values, we end up with two points

$$(3, 0), \qquad (0, \tfrac{\pi}{2})$$

  The tangents are indeed vertical here, because $\frac{dy}{d\theta} \neq 0$ at these points.

□

Section 10.4

21. Find the area of the region enclosed by one loop of the curve $r = 1 + 2\sin\theta$ (inner loop).

Solution - We'd like to use the formula $A = \int_\alpha^\beta \frac{1}{2}r^2\, d\theta$, but we first need to figure out what the angles α and β should be. To do this, it's best to start with a quick sketch of this curve.

Rather than plotting individual points, plot r as a function of θ in Cartesian coordinates (i.e. the standard way we plot functions), then use this sketch to help sketch the curve. (To check your progress on this, use Wolfram Alpha. See Examples 7 and 8 on page 658.)

The inner loop starts and ends at the origin, i.e. $r = 0$. Setting $r = 0$, we have

$$1 + 2\sin\theta = 0 \implies \sin\theta = -\tfrac{1}{2} \implies \theta = \tfrac{7\pi}{6} \text{ or } \tfrac{11\pi}{6}$$

Comparing with the sketch, we see that the inner loop ranges from $\theta = \frac{7\pi}{6}$ to $\theta = \frac{11\pi}{6}$. Now

$$\begin{aligned}
A &= \int_{7\pi/6}^{11\pi/6} \tfrac{1}{2}(1+2\sin\theta)^2 \, d\theta \\
&= \tfrac{1}{2}\int_{7\pi/6}^{11\pi/6} 1 + 4\sin\theta + 4\sin^2\theta \, d\theta \\
&= \tfrac{1}{2}\int_{7\pi/6}^{11\pi/6} 3 + 4\sin\theta - 2\cos 2\theta \, d\theta \\
&= \tfrac{1}{2}(3\theta - 4\cos\theta - \sin 2\theta)\Big|_{\theta=7\pi/6}^{11\pi/6} \\
&= \tfrac{1}{2}[(\tfrac{11\pi}{2} - 4\cos(\tfrac{11\pi}{6}) - \sin(\tfrac{11\pi}{3})) - (\tfrac{7\pi}{2} - 4\cos(\tfrac{7\pi}{6}) - \sin(\tfrac{7\pi}{3}))] \\
&= \tfrac{1}{2}(2\pi - 8\cos(\tfrac{\pi}{6}) + 2\sin(\tfrac{\pi}{3})) \\
&= \pi - 4\cos(\tfrac{\pi}{6}) + \sin(\tfrac{\pi}{3}) \\
&= \pi - 2\sqrt{3} + \tfrac{\sqrt{3}}{2} \\
&= \pi - \tfrac{3\sqrt{3}}{2}
\end{aligned}$$

□

27. Find the area of the region that lies inside the first curve and outside the second curve.

$$r = 3\cos\theta, \qquad r = 1 + \cos\theta$$

<u>Solution</u> - I'll leave it to you to sketch these curves as an exercise. (Use Wolfram Alpha if you need help.) To find the area of the region, find the θ-values corresponding to the points of intersection of the two curves. Set

$$3\cos\theta = 1 + \cos\theta \implies 2\cos\theta = 1 \implies \cos\theta = \tfrac{1}{2} \implies \theta = \tfrac{\pi}{3} \text{ or } \tfrac{5\pi}{3}$$

Looking at the diagram, we want the region $\theta \leq \frac{\pi}{3}$ or $\theta \geq \frac{5\pi}{3}$. Another reason to write this is $-\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3}$. Write

$$\begin{aligned}
A &= \int_{-\pi/3}^{\pi/3} \tfrac{1}{2}(3\cos\theta)^2 \, d\theta - \int_{-\pi/3}^{\pi/3} \tfrac{1}{2}(1+\cos\theta)^2 \, d\theta \\
&= \tfrac{1}{2}\int_{-\pi/3}^{\pi/3} 9\cos^2\theta - (1+\cos\theta)^2 \, d\theta \\
&= \int_{0}^{\pi/3} 9\cos^2\theta - (1+\cos\theta)^2 \, d\theta \qquad \text{(integrand is even)} \\
&= \int_{0}^{\pi/3} 8\cos^2\theta - 2\cos\theta - 1 \, d\theta \\
&= \int_{0}^{\pi/3} 3 + 4\cos 2\theta - 2\cos\theta \, d\theta \quad \text{(half-angle formula)} \\
&= (3\theta + 2\sin 2\theta - 2\sin\theta)\Big|_{\theta=0}^{\pi/3} \\
&= \pi + 2\sin\tfrac{2\pi}{3} - 2\sin\tfrac{\pi}{3} \\
&= \pi
\end{aligned}$$

□

41. Find all points of intersection of the given curves.

$$r = \sin\theta, \qquad r = \sin 2\theta$$

Solution - To find the points of intersection, we set $\sin\theta = \sin 2\theta$. We have

$$\sin\theta - \sin 2\theta = 0 \implies \sin\theta - 2\sin\theta\cos\theta = 0 \implies \sin\theta(1 - 2\cos\theta) = 0$$

Thus, $\sin\theta = 0$ or $\cos\theta = \frac{1}{2}$. This gives rise to the values $\theta = 0$, $\theta = \pi$, $\theta = \frac{\pi}{3}$, $\theta = \frac{5\pi}{3}$. In polar coordinates, the points of intersection are

$$(0,0), \quad (0,\pi), \quad (\tfrac{\sqrt{3}}{2}, \tfrac{\pi}{3}), \quad (-\tfrac{\sqrt{3}}{2}, \tfrac{5\pi}{3})$$

We can write these a little more nicely. Note that $(0,0)$ and $(0,\pi)$ both correspond to the origin. Moreover, we can rewrite the fourth point as $(\frac{\sqrt{3}}{2}, \frac{2\pi}{3})$. Thus, there are *three* points of intersection, namely

$$(0,0), \qquad (\tfrac{\sqrt{3}}{2}, \tfrac{\pi}{3}), \qquad (\tfrac{\sqrt{3}}{2}, \tfrac{2\pi}{3})$$

□

45. Find the exact length of the polar curve $r = 2\cos\theta$, $0 \le \theta \le \pi$.

Solution

$$\begin{aligned} L &= \int_\alpha^\beta \sqrt{r^2 + (\tfrac{dr}{d\theta})^2}\, d\theta \\ &= \int_0^\pi \sqrt{(2\cos\theta)^2 + (-2\sin\theta)^2}\, d\theta \\ &= \int_0^\pi \sqrt{4(\cos^2\theta + 4\sin^2\theta)}\, d\theta \\ &= \int_0^\pi 2\, d\theta \\ &= 2\pi \end{aligned}$$

If you graph this curve, you'll see that it's a circle of radius 1 (and centre (1,0)). That being the case, it shouldn't be surprising that the length of the curve is 2π, i.e. the circumference of the circle. □