21-122 - Week 13, Recitation 2

Agenda

- Evaluating Limits with Taylor Series
- Section 11.10: 65, 67, 69, Example 13
- Section 11.11: 5, 9

Evaluating Limits with Taylor Series

Evaluate $\lim_{x\to 0} \frac{xe^{x^2}}{\sin x}$ using Taylor series.

$$\begin{aligned}\lim_{x\to 0} \frac{xe^{x^2}}{\sin x} &= \lim_{x\to 0} \frac{x(1+x^2+\frac{x^4}{4!}+\cdots)}{x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots}\\ &= \lim_{x\to 0} \frac{1+x^2+\frac{x^4}{4!}+\cdots}{1-\frac{x^2}{3!}+\frac{x^4}{5!}-\cdots}\\ &= \frac{1+0+0+\cdots}{1-0+0-\cdots}\\ &= 1\end{aligned}$$

Evaluate $\lim_{x\to 1} \frac{\ln x}{1-x}$. (Hint: $\ln x = \ln(1+(x-1))$ using Taylor series.

$$\begin{aligned}\lim_{x\to 1} \frac{\ln x}{1-x} &= \lim_{x\to 0} \frac{\ln(1+(x-1))}{1-x}\\ &= \lim_{x\to 1} \frac{(x-1)+\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}+\cdots}{1-x}\\ &= -\lim_{x\to 1} \frac{(x-1)+\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}+\cdots}{x-1}\\ &= -\lim_{x\to 1}(1+\frac{(x-1)}{2}+\frac{(x-1)^2}{3}+\cdots)\\ &= -(1+0+0+\cdots)\\ &= -1\end{aligned}$$

Section 11.10

65. Find the sum of the series.

$$(65)\quad \sum_{n=1}^{\infty}(-1)^{n-1}\frac{3^n}{n5^n}, \qquad (67)\quad \sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n+1}}{4^{2n+1}(2n+1)!} \qquad (69)\quad 3+\frac{9}{2!}+\frac{27}{3!}+\frac{81}{4!}+\cdots$$

Solution - We use Table 1 on page 762 of the textbook. For (65), we have $\ln(1+\frac{3}{5}) = \ln\frac{8}{5}$. For (67), we have $\sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$. We can rewrite (69) as follows:

$$\begin{aligned}3+\frac{9}{2!}+\frac{27}{3!}+\frac{81}{4!}+\cdots &= \sum_{n=1}^{\infty}\frac{3^n}{n!}\\ &= \sum_{n=0}^{\infty}\frac{3^n}{n!}-1\\ &= e^3-1\end{aligned}$$

□

Example 13: Find the first three nonzero terms in the Maclaurin series for

(a) $e^x \sin x$

(b) $\tan x$

Solution - For both questions, use Table 1. For (a),

$$\begin{aligned} e^x \sin x &= (1 + x + \tfrac{x^2}{2!} + \tfrac{x^3}{3!} + \cdots)(x - \tfrac{x^3}{3!} + \cdots) \\ &= 1 \cdot x + x \cdot x + (\tfrac{x^2}{2!} \cdot x - 1 \cdot \tfrac{x^3}{3!}) + \cdots \\ &= x + x^2 + \tfrac{1}{6}x^3 + \cdots \end{aligned}$$

(b) Write $\tan x = \frac{\sin x}{\cos x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots}$. To find the first terms of the Maclaurin series, we do long division. We get

$$\tan x = x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \cdots$$

(for more details, see textbook). □

Section 11.11

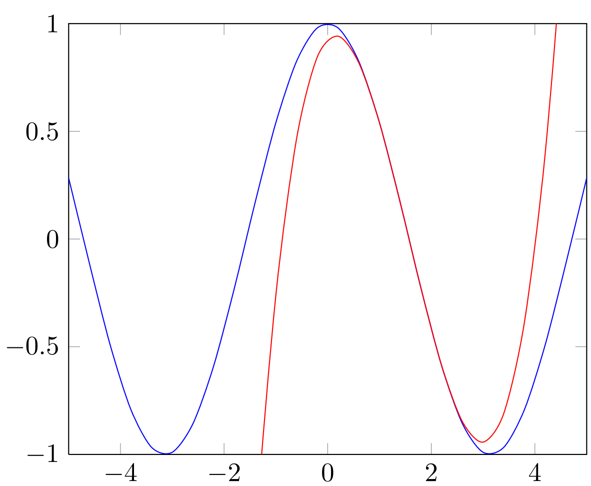
5, 9. Find the Taylor polynomial $T_3(x)$ for f centered at a. Graph f and T_3.

$$(5)\ f(x) = \cos x,\ a = \tfrac{\pi}{2}, \qquad (9)\ f(x) = xe^{-2x},\ a = 0$$

Solution - (5) We have

$$f(x) = \cos x, \quad f'(x) = -\sin x, \quad f''(x) = -\cos x, \quad f'''(x) = \sin x$$

$$\begin{aligned} T_3(x) &= f(\tfrac{\pi}{2}) + f'(\tfrac{\pi}{2})(x - \tfrac{\pi}{2}) + \frac{f''(\frac{\pi}{2})}{2!}(x - \tfrac{\pi}{2})^2 + \frac{f'''(\frac{\pi}{2})}{3!}(x - \tfrac{\pi}{2})^3 \\ &= -(x - \tfrac{\pi}{2}) + \tfrac{1}{6}(x - \tfrac{\pi}{2})^3 \end{aligned}$$



(9) We do this in two ways. Here's a solution by hand.

$$f(x) = xe^{-2x}, \quad f'(x) = (1 - 2x)e^{-2x}, \quad f''(x) = (4x - 4)e^{-2x}, \quad f'''(x) = (12 - 8x)e^{-2x}$$

Now

$$\begin{aligned} T_3(x) &= f(0) + f'(0)x + \tfrac{f''(0)}{2!}x^2 + \tfrac{f'''(0)}{3!}x^3 \\ &= x - 2x^2 + 2x^3 \end{aligned}$$

Alternately, let's use Table 1 to write

$$f(x) = xe^{-2x} = x(1 + (-2x) + \tfrac{(-2x)^2}{2!} + \cdots) = x - 2x^2 + 2x^3 + \cdots \implies T_3(x) = x - 2x^2 + 2x^3$$

I had technical issues trying to plot this. To see the graphs, go to Wolfram Alpha and enter "plot xexp(-2x), x - 2x**2 + 2x**3".