21-122 - Week 13, Recitation 1

Agenda

- Review
- Section 11.10: 19, 27, 33, 57, 63
- HW 9 Due

Section 11.10

19. Find the Taylor series for $f(x) = \cos x$ centered at $a = \pi$. Also find the associated radius of convergence.
Solution - Write

$$\begin{aligned} f(x) &= \cos x & f(\pi) &= -1 \\ f'(x) &= -\sin x & f'(\pi) &= 0 \\ f''(x) &= -\cos x & f''(\pi) &= 1 \\ f'''(x) &= \sin x & f'''(\pi) &= 0 \\ f^{(4)}(x) &= \cos x & f^{(4)}(\pi) &= -1 \end{aligned}$$

The derivatives repeat in a cycle of four, so we can write

$$\begin{aligned} f(x) &= f(\pi) + \tfrac{f'(\pi)}{1!}(x-\pi) + \tfrac{f''(\pi)}{2!}(x-\pi)^2 + \tfrac{f'''(\pi)}{3!}(x-\pi)^3 + \cdots \\ &= -1 + \tfrac{(x-\pi)^2}{2!} - \tfrac{(x-\pi)^4}{4!} + \tfrac{(x-\pi)^6}{6!} - \cdots \\ &= \sum_{n=0}^{\infty} (-1)^{n+1} \tfrac{(x-\pi)^{2n}}{(2n)!} \end{aligned}$$

For the radius of convergence, we use the Ratio Test, $a_n = (-1)^{n+1}\frac{(x-\pi)^{2n}}{(2n)!}$. For $x \neq \pi$, we have

$$\lim_{n\to\infty} \left|\tfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\tfrac{(x-\pi)^{2n+2}}{(x-\pi)^{2n}} \cdot \tfrac{(2n)!}{(2n+2)!}\right| = \lim_{n\to\infty} \left|\tfrac{(x-\pi)^2}{(2n+2)(2n+1)}\right| = 0$$

Thus, the radius of convergence is $R = \infty$. □

27. Use the binomial series to expand the function $\frac{1}{(2+x)^3}$ as a power series. State the radius of convergence.
Solution - Write

$$\tfrac{1}{(2+x)^3} = (2+x)^{-3} = 2^{-3}(1+\tfrac{x}{2})^{-3} = 2^{-3}\sum_{n=0}^{\infty}\binom{-3}{n}(\tfrac{x}{2})^n$$

We can simplify this by writing

$$\binom{-3}{n}(\tfrac{x}{2})^n = \tfrac{(-3)(-3-1)(-3-2)\cdots(-3-n+1)}{n!}\tfrac{x^n}{2^n} = (-1)^n \tfrac{3\cdot4\cdot5\cdots(3+n-1)}{n!}\tfrac{x^n}{2^n} = (-1)^n\tfrac{(n+2)!}{2\cdot n!}\tfrac{x^n}{2^n} = (-1)^n\tfrac{(n+1)(n+2)}{2^{n+1}}x^n$$

Now

$$f(x) = 2^{-3}\sum_{n=0}\binom{-3}{n}(\tfrac{x}{2})^n = \sum_{n=0}^{\infty}(-1)^n\tfrac{(n+1)(n+2)}{2^{n+4}}x^n$$

This expansion is only valid when $|\frac{x}{2}| < 1$, or $|x| < 2$. Thus, the radius of convergence is $R = 2$. □

33. Use a Maclaurin series in Table 1 to obtain the Maclaurin series for $f(x) = x\cos(\frac{1}{2}x^2)$.
Solutions - Write

$$x\cos(\tfrac{1}{2}x^2) = x\sum_{n=0}^{\infty}(-1)^n\frac{(\frac{1}{2}x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+1}}{2^{2n}(2n)!}$$

□

57. Use series to evaluate the limit $\lim_{x\to 0}\frac{\sin x - x + \frac{1}{6}x^3}{x^5}$.
Solution - Write

$$\begin{aligned}\lim_{x\to 0}\frac{\sin x - x + \frac{1}{6}x^3}{x^5} &= \lim_{x\to 0}\frac{(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots) - x + \frac{1}{6}x^3}{x^5}\\ &= \lim_{x\to 0}\frac{\frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots}{x^5}\\ &= \lim_{x\to 0}(\tfrac{1}{5!} - \tfrac{x^2}{7!} + \tfrac{x^4}{9!} - \cdots)\\ &= \tfrac{1}{5!}\\ &= \tfrac{1}{120}\end{aligned}$$

□

63. Find the sum of the series $\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n}}{n!}$.
Solution - We can rewrite this as

$$\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n}}{n!} = \sum_{n=0}^{\infty}\frac{(-x^4)^n}{n!} = e^{-x^4}$$

□