21-122 - Week 12, Recitation 1

Agenda

- Return HW 8, Quiz 3
- Quiz 3 Solutions
- 11.7 (Strategy for Testing Series) - 7, 18, 33
- 11.8 (Power Series) - 28
- 11.9 (Functions as Power Series) - 8, 22, 24

Section 11.7

Test the series for convergence or divergence.

$$(7)\ \sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}, \qquad (18)\ \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}-1}, \qquad (33)\ \sum_{n=1}^{\infty} (\tfrac{n}{n+1})^{n^2}$$

Solutions - For (7), let's use the Integral Test. The function $f(x) = \frac{1}{x\sqrt{\ln x}}$ is clearly continuous, positive, and decreasing for $x \geq 2$ (since x and $\ln x$ are increasing). Write

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\ dx \overset{(u=\ln x)}{=} \int_{\ln 2}^{\infty} \frac{1}{\sqrt{u}}\ du$$

The integral above is known to diverge (recall that $\int_1^{\infty} \frac{1}{x^p}\ dx$ diverges for $p \leq 1$). Therefore, the series diverges.

Since (18) is an alternating series, the Alternating Series Test seems appropriate. Write $b_n = \frac{1}{\sqrt{n}-1}$. Since $\sqrt{n}-1$ is increasing for $n \geq 2$, then b_n is decreasing. Moreover, $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}-1} = 0$. By the Alternating Series Test, the series converges.

Since the terms in (33) are raised to the power n^2, the Root Test seems like it might be helpful. Write $a_n = (\frac{n}{n+1})^{n^2}$. We have

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left((\tfrac{n}{n+1})^{n^2}\right)^{1/n} = \lim_{n\to\infty} (\tfrac{n}{n+1})^n$$

This is an indeterminate form (of type 1^{∞}), so use L'Hopital's Rule to determine the limit. Write

$$\begin{aligned}
\lim_{n\to\infty} \log((\tfrac{n}{n+1})^n) &= \lim_{n\to\infty} n\log(\tfrac{n}{n+1}) \\
&= \lim_{x\to\infty} x\log(\tfrac{x}{x+1}) \\
&= \lim_{x\to\infty} \frac{\log(\frac{x}{x+1})}{\frac{1}{x}} \\
&= \lim_{x\to\infty} \frac{\frac{1}{x}-\frac{1}{x+1}}{-\frac{1}{x^2}} \quad \text{(L'Hopital)} \\
&= \lim_{x\to\infty} -x^2(\tfrac{1}{x}-\tfrac{1}{x+1}) \\
&= \lim_{x\to\infty} -\cdot \frac{x^2}{x(x+1)} \\
&= -1
\end{aligned}$$

Thus, $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} (\frac{n}{n+1})^n = e^{-1} < 1$, so by the Root Test, the series convergs. $\square$

Section 11.8

(28) Find the radius and interval of convergence for the series $\sum\limits_{n=1}^{\infty} \frac{n!x^n}{1\cdot 3\cdot 5\cdots(2n-1)}$.
Solution - For $x = 0$, we have convergence. For $x \neq 0$, use the Ratio Test.

$$\lim_{n\to\infty} |\tfrac{a_{n+1}}{a_n}| = \lim_{n\to\infty} |\frac{\frac{(n+1)!x^{n+1}}{1\cdot 3\cdot 5\cdots(2(n+1)-1)}}{\frac{n!x^n}{1\cdot 3\cdot 5\cdots(2n-1)}}| = \lim_{n\to\infty} |\tfrac{(n+1)!}{n!}\tfrac{x^{n+1}}{x^n}\tfrac{1\cdot 3\cdot 5\cdots(2n-1)}{1\cdot 3\cdot 5\cdots(2(n+1)-1)}| = \lim_{n\to\infty} \tfrac{n}{2(n+1)-1}|x| = \tfrac{1}{2}|x|$$

We know the series converges if $\frac{1}{2}|x| < 1$, i.e. $|x| < 2$. The series diverges if $\frac{1}{2}|x| > 1$, i.e. $|x| > 2$. Thus, the radius of convergence is $R = 2$.

To determine the interval of convergence, we need to test the endpoints $x = 2$ and $x = -2$.

- When $x = 2$, the terms of the series can be written as

$$a_n = \tfrac{n!2^n}{1\cdot 3\cdot 5\cdots(2n-1)} = \tfrac{2\cdot 4\cdot 6\cdots(2n)}{1\cdot 3\cdot 5\cdots(2n-1)} = \tfrac{2}{1}\tfrac{4}{3}\tfrac{6}{5}\cdots\tfrac{2n}{(2n-1)} > 1$$

  In particular, $\lim\limits_{n\to\infty} a_n \neq 0$, so by the Divergence Test, the series diverges.

- When $x = -2$, we have the alternating series $\sum\limits_{n=1}^{\infty}(-1)^n a_n$, where a_n is the same as in the previous case. But from the previous case, we know that $\lim\limits_{n\to\infty} a_n \neq 0$, so $\lim\limits_{n\to\infty}(-1)^n a_n \neq 0$. By the Divergence Test again, the series diverges.

Therefore, the interval of convergence is $(-2, 2)$. $\square$

Section 11.9

Find a power series representation and the radius/interval of convergence (only for (8)) for the following functions.

(8) $f(x) = \frac{x}{2x^2+1}$ (22) $f(x) = \ln(x^2+4)$ (24)$f(x) = \tan^{-1}(2x)$

Solutions - For (8), write

$$f(x) = \tfrac{x}{2x^2+1} = x\cdot\tfrac{1}{1-(-2x^2)} = x\sum_{n=0}^{\infty}(-2x^2)^n = x\sum_{n=0}^{\infty}(-2)^n x^{2n} = \sum_{n=0}^{\infty}(-2)^n x^{2n+1}$$

The series converges exactly when $|-2x^2| < 1$, or when $x^2 < \frac{1}{2}$, i.e. $|x| < \frac{1}{\sqrt{2}}$. Thus, the radius of convergence is $R = \frac{1}{\sqrt{2}}$ and the interval of convergence is $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

(22) The derivative of $f(x)$ is $f'(x) = \frac{2x}{x^2+4}$. To find a power series for $f(x)$, let's find a power series for its derivative and then integrate. Write

$$f'(x) = \tfrac{2x}{x^2+4} = \tfrac{2x}{4}\cdot\tfrac{1}{1-(-\frac{x^2}{4})} = \tfrac{x}{2}\sum_{n=0}^{\infty}(-\tfrac{x^2}{4})^n = \tfrac{x}{2}\sum_{n=0}^{\infty}(-\tfrac{1}{4})^n x^{2n} = \sum_{n=0}^{\infty}\tfrac{1}{2}(-\tfrac{1}{4})^n x^{2n+1}$$

Now we can write

$$f(x) = \ln(x^2+4) = \int\sum_{n=0}^{\infty}\tfrac{1}{2}(-\tfrac{1}{4})^n x^{2n+1}\,dx = \sum_{n=0}^{\infty}\tfrac{1}{2}(-\tfrac{1}{4})^n\int x^{2n+1}\,dx = \sum_{n=0}^{\infty}\tfrac{1}{2}(-\tfrac{1}{4})^n\tfrac{x^{2n+2}}{2n+2} + C$$

To find C, substitute $x = 0$. We get $f(0) = \ln(4) = C$, thus

$$\ln(x^2+4) = \sum_{n=0}^{\infty} \tfrac{1}{2}(-\tfrac{1}{4})^n \tfrac{x^{2n+2}}{2n+2} + \ln(4)$$

We approach (24) similarly to (22). Write $f'(x) = \frac{1}{1+(2x)^2} \cdot 2 = \frac{2}{1+4x^2}$. Now

$$f'(x) = 2\tfrac{1}{1-(-4x^2)} = 2\sum_{n=0}^{\infty}(-4x^2)^n = 2\sum_{n=0}^{\infty}(-4)^n x^{2n} = \sum_{n=0}^{\infty}(-1)^n 2^{2n+1} x^{2n}$$

Now

$$f(x) = \int f'(x)\,dx = \int \sum_{n=0}^{\infty}(-1)^n 2^{2n+1} x^{2n}\,dx = \sum_{n=0}^{\infty}(-1)^n 2^{2n+1}\int x^{2n}\,dx = \sum_{n=0}^{\infty}(-1)^n 2^{2n+1}\tfrac{x^{2n+1}}{2n+1} + C$$

Since $f(0) = \tan^{-1}(0) = 0$, then $C = 0$, so

$$f(x) = \sum_{n=0}^{\infty}(-1)^n 2^{2n+1}\tfrac{x^{2n+1}}{2n+1}$$

$\square$