21-122 - Week 1, Recitation 2

Agenda

- Announcements: Office hours
- Review
- Even and odd functions
- 7.1: 15, 17, 19, 24, 40, 41, Example 6

Review

- Recall - We say $f(x)$ is *even* if $f(-x) = f(x)$ for all x domain(f). We say $f(x)$ is *odd* if $f(-x) = -f(x)$ for all x in the domain(f).
- Theorem 7 on page 412: Suppose f is continuous on $[-a, a]$.
  (a) If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.
  (b) If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.
- Integration by Parts (IBP):

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx \quad \text{or} \quad \int u\,dv = uv - \int v\,du$$

For definite integrals, we have

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)\,dx$$

Even and Odd Functions

We begin by listing some properties of even and odd functions.

1. The product of two even functions is even.
2. The product of two odd functions is even.
3. The product of an even function and an odd function is odd.

Proof

1. Let $f(x), g(x)$ be even functions. Want to show fg is even. For any x in domain(fg), we have

$$(fg)(-x) = f(-x)g(-x) = f(x)g(x) = (fg)(x)$$

2. Let $f(x), g(x)$ be odd functions. Want to show fg is even. For any x in domain(fg), we have

$$(fg)(-x) = f(-x)g(-x) = (-f(x))(-g(x)) = f(x)g(x) = (fg)(x)$$

3. Let $f(x)$ be even, $g(x)$ odd. Want to show fg is odd. For any x in domain(fg), we have

$$(fg)(-x) = f(-x)g(-x) = f(x)(-g(x)) = -f(x)g(x) = -(fg)(x)$$

□

Examples

- x and $\sin x$ are odd, so $x \sin x$ is even.
- $\sin x$ is odd, $\cos x$ is even, so $\sin x \cos x$ is odd.
- x^2 and $|x|$ are even, so $|x|x^2$ is even.

More properties of even and odd functions...

1. The sum of two even functions is even.
2. The sum of two odd functions is odd.

Proof

1. Let $f(x)$, $g(x)$ be even. For any x in domain$(f+g)$, we have

$$(f+g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f+g)(x)$$

2. Let $f(x)$, $g(x)$ be odd. For any x in domain$(f+g)$, we have

$$(f+g)(-x) = f(-x) + g(-x) = -f(x) - g(x) = -(f(x) + g(x)) = -(f+g)(x)$$

□

Examples - By Theorem 7,

$$\int_{-10}^{10} (x - 5x^3 + \tfrac{\sin x}{(1+x^2)})\, dx = 0$$

$$\int_{-1}^{1} (x^2 + \cos x)\, dx = 2\int_{0}^{1} (x^2 + \cos x)\, dx$$

Section 7.1 - Integration by Parts

(15) $\int (\ln x)^2\, dx$.
Solution - Write $u = (\ln x)^2$, $dv = dx$, so then $du = 2\ln x \cdot \frac{1}{x}dx$ and $v = x$. Now

$$\int (\ln x)^2\, dx = (\ln x)^2 - \int x(2\ln x \cdot \tfrac{1}{x}dx) = (\ln x)^2 - 2\int \ln x\, dx \qquad (*)$$

To evaluate $\int \ln x\, dx$, choose $u = \ln x$, $dv = dx$, so then $du = \frac{1}{x}\, dx$ and $v = x$. Then

$$\int \ln x\, dx = x\ln x - \int x\tfrac{1}{x}\, dx = x\ln x - \int\, dx = x\ln x - x + C$$

Substituting this into $(*)$, we have

$$\int (\ln x)^2\, dx = (\ln x)^2 - 2(x\ln x - x + C) = (\ln x)^2 - 2x\ln x + 2x + C,$$

where we have absorbed the -2 into C. □

(17) $\int e^{-3\theta}\cos(5\theta)\,d\theta$

Solution - Write $u = e^{-3\theta}$, $dv = \cos(5\theta)\,d\theta$, so then $du = -3e^{-3\theta}d\theta$ and $v = \frac{1}{5}\sin(5\theta)$. Now

$$\begin{aligned}\int e^{-3\theta}\cos(5\theta)\,d\theta &= \tfrac{1}{5}e^{-3\theta}\sin(5\theta) - \int \tfrac{1}{5}\sin(5\theta)(-3)e^{-3\theta}\,d\theta \\ &= \tfrac{1}{5}e^{-3\theta}\sin(5\theta) + \tfrac{3}{5}\int e^{-3\theta}\sin(5\theta)\,d\theta \quad (*)\end{aligned}$$

Continue by applying IBP to the integral in $(*)$. Choose $u = e^{-3\theta}$, $dv = \sin(5\theta)\,d\theta$, so then $du = -3e^{-3\theta}d\theta$ and $v = -\frac{1}{5}\cos(5\theta)$. Now

$$\int e^{-3\theta}\sin(5\theta)\,d\theta = -\tfrac{1}{5}e^{-3\theta}\cos(5\theta) - \tfrac{3}{5}\int e^{-3\theta}\cos(5\theta)\,d\theta \qquad (**)$$

Substituting $(**)$ into $(*)$, we have

$$\begin{aligned}\int e^{-3\theta}\cos(5\theta)\,d\theta &= \tfrac{1}{5}e^{-3\theta}\sin(5\theta) + \tfrac{3}{5}(-\tfrac{1}{5}e^{-3\theta}\cos(5\theta) - \tfrac{3}{5}\int e^{-3\theta}\cos(5\theta)\,d\theta) \\ &= \tfrac{1}{5}e^{-3\theta}\sin(5\theta) - \tfrac{3}{25}e^{-3\theta}\cos(5\theta) - \tfrac{9}{25}\int e^{-3\theta}\cos(5\theta)\,d\theta) \\ \implies (1+\tfrac{9}{25})\int e^{-3\theta}\cos(5\theta)\,d\theta &= \tfrac{1}{5}e^{-3\theta}\sin(5\theta) - \tfrac{3}{25}e^{-3\theta}\cos(5\theta) \\ \implies \int e^{-3\theta}\cos(5\theta) &= \tfrac{5}{34}e^{-3\theta}\sin(5\theta) - \tfrac{3}{34}e^{-3\theta}\cos(5\theta) + C \\ &= \tfrac{1}{34}e^{-3\theta}(5\sin(5\theta) - 3\cos(5\theta)) + C\end{aligned}$$

□

(19) $\int z^3e^z\,dz$.

Solution - Write $u = z^3$, $dv = e^z\,dz$, so $du = 3z^2\,dz$ and $v = e^z$. Now

$$\int z^3e^z\,dz = z^3e^z - 3\int z^2e^z\,dz \quad (*)$$

To evaluate $\int z^2e^z\,dz$, write $u = z^2$, $dv = e^z\,dz$, so then $du = 2z\,dz$ and $v = e^z$. Now

$$\int z^2e^z\,dz = z^2e^z - 2\int ze^z\,dz \quad (**)$$

Finally, to evaluate $\int ze^z\,dz$, write $u = z$, $dv = e^z\,dz$, so then $du = dz$ and $v = e^z$. Now

$$\int ze^z\,dz = ze^z - \int e^z\,dz = ze^z - e^z + C \quad (***)$$

Substituting $(***)$ into $(**)$, we get

$$\int z^2e^z\,dz = z^2e^z - 2(ze^z - e^z + C) = e^z(z^2 - 2z + 2) + C$$

Substituting this result into $(*)$, we have

$$\int z^3e^z\,dz = z^3e^z - 3(e^z(z^2 - 2z + 2) + C) = e^z(z^3 - 3z^2 + 6z - 6) + C$$

□

(24) $\int_0^1 (x^2+1)e^{-x}\,dx$

Solution - Write $u = x^2 + 1$, $dv = e^{-x}\,dx$, so then $du = 2x\,dx$ and $v = -e^{-x}$. Now

$$\begin{aligned}\int_0^1 (x^2+1)e^{-x}\,dx &= (x^2+1)(-e^{-x})\Big|_0^1 - \int_0^1 2x(-e^{-x})\,dx \\ &= 1 - 2e^{-1} + 2\int_0^1 xe^{-x}\,dx\end{aligned}$$

To evaluate $\int_0^1 xe^{-x}\,dx$, write $u = x$, $dv = e^{-x}\,dx$, so then $du = dx$ and $v = -e^{-x}$. Now

$$\int_0^1 xe^{-x}\,dx = x(-e^{-x})\Big|_0^1 - \int_0^1 -e^{-x}\,dx = 0 - e^{-1} - e^{-x}\Big|_0^1 = -e^{-1} - (e^{-1} - 1) = 1 - 2e^{-1}$$

Thus,

$$\int_0^1 (x^2+1)e^{-x}\,dx = 1 - 2e^{-1} + 2(1 - 2e^{-1}) = 3(1 - 2e^{-1})$$

□

(40) First make a substitution and then use IBP to evaluate $\int_0^\pi e^{\cos t}\sin 2t\,dt$.
Solution - If we write $\sin 2t = 2\sin t\cos t$ and substitute $w = \cos t$, then $dw = -\sin t\,dt$ and then

$$\int_0^\pi e^{\cos t}\sin 2t\,dt = 2\int_0^\pi e^{\cos t}\cos t\sin t\,dt = 2\int_1^{-1} e^w w(-dw) = 2\int_{-1}^1 we^w\,dw$$

Now using IBP with $u = w$ and $dv = e^w\,dw$, then $du = dw$ and $v = e^w$, so

$$2\int_{-1}^1 ue^u\,du = 2(ue^u\Big|_{-1}^1 - \int_{-1}^1 e^u\,du) = 2(e + e^{-1} - e^u\Big|_{-1}^1) = 2(e + e^{-1} - (e - e^{-1}) = 4e^{-1}$$

□

(41) First make a substitution and then use IBP to evaluate $\int x\ln(1+x)\,dx$.
Solution - Substitute $w = 1 + x$, so $dw = dx$. Then

$$\int x\ln(1+x)\,dx = \int (w-1)\ln w\,dw$$

Now use IBP. Write $u = \ln w$ and $dv = (w-1)\,dw$, so then $du = \frac{1}{w}\,dw$ and $v = \frac{w^2}{2} - w$. Now

$$\begin{aligned}\int x\ln(1+x)\,dx &= \int (w-1)\ln w\,dw \\ &= (\tfrac{w^2}{2} - w)\ln w - \int (\tfrac{w^2}{2} - w)\tfrac{1}{w}\,dw \\ &= (\tfrac{w^2}{2} - w)\ln w - \int (\tfrac{w}{2} - 1)\,dw \\ &= (\tfrac{w^2}{2} - w)\ln w - \tfrac{w^2}{4} + w + C \\ &= (\tfrac{(1+x)^2}{2} - (1+x))\ln(1+x) - \tfrac{(1+x)^2}{4} + (1+x) + C \\ &= \tfrac{1}{2}(x^2-1)\ln(1+x) + \tfrac{1}{2}x + \tfrac{1}{4}x^2 + C\end{aligned}$$

□

Example 6: Prove the reduction formula

$$\int \sin^n x\,dx = -\tfrac{1}{n}\cos x\sin^{n-1}x + \tfrac{n-1}{n}\int \sin^{n-2}x\,dx,$$

where $n \geq 2$ is an integer.
Solution - Write $u = \sin^{n-1} x$, $dv = \sin x \; dx$, so then $du = (n-1)\sin^{n-2} x \cos x \; dx$ and $v = -\cos x$. Now, using the fact that $\cos^2 x = 1 - \sin^2 x$, we have

$$\begin{aligned} \int \sin^n x \; dx &= -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x \cos^2 x \; dx \\ &= -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x \; dx - (n-1) \int \sin^n x \; dx \\ \implies n \int \sin^n x \; dx &= -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x \; dx \\ \implies \int \sin^n x \; dx &= -\tfrac{1}{n} \cos x \sin^{n-1} x + \tfrac{n-1}{n} \int \sin^{n-2} x \; dx \end{aligned}$$

□